Demonstrating that VaR (for worse enough outcomes) is a coherent risk measure for a Gaussian, i.e. multi-variate normal, distribution

[Nematrian website page: ValueAtRiskCoherentForGaussian, © Nematrian 2015]

For a risk measure to be coherent it must satisfy:

(a) **Subadditivity**: for any pair of loss variables, x_1 and x_2

$$r(x_1 + x_2) \leq r(x_1) + r(x_2)$$

(b) **Monotonicity**: if, for all states of the world, $x_1 > x_2$ then

$$r(x_1) \geq r(x_2)$$

(c) **Homogeneity**: for any constant $\lambda > 0$ and random loss variable x

$$r(\lambda x_1) = \lambda r(x_1)$$

(d) **Translational invariance**: for any constant d and random loss variable x

$$r(x + d) = r(x) + d$$

For a normal distribution the VaR at the α confidence level is as follows, if the distribution (for the given variable of interest) is distributed $N(\mu, \sigma^2)$:

$$VaR = \mu + \sigma N^{-1}(1 - \alpha)$$

Homogeneity and translational invariance therefore immediately apply.

Subadditivity holds (as long as $N^{-1}(1 - \alpha) > 0$, i.e. $\alpha > 0.5$) because the standard deviation of the sum of two random variables is less than or equal to the sum of their standard deviations

Monotonicity holds because Normally distributed random variables have positive support on the real line, so $x_1 > x_2$ in all states of the world only if x_1 and x_2 are perfectly correlated, in which case $x_1 = x_2 + c$, where c is constant, hence $VaR(x_1) = VaR(x_1) + c$ for any α.