Calculating TVaR if a distribution has a quantile-quantile form (versus the normal distribution) that is a cubic polynomial

[Nematrian website page: TVaRForCubicQuantileQuantileRelationships, © Nematrian 2015]

A method that some practitioners use to estimate risk measures such as Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) for fat-tailed distributions is make use of the Cornish-Fisher asymptotic expansion, see derivation of the Cornish-Fisher expansion or MnCornishFisher4. This is a methodology for predicting the shape of a (univariate) distributional form merely from the moments of the distribution, most commonly merely its mean, standard deviation, skew and (excess) kurtosis.

However, Kemp (2009) notes that the Cornish-Fisher approach has some undesirable features including not necessarily giving appropriate weight to different parts of the distributional form. In effect it can result in estimation of outlying quantiles of the distribution more from the distributional shape in the centre of the distribution than from its shape in its tails, which is counterintuitive and liable to error. Kemp proposes a more empirical approach in which the distributional form and hence the risk measure is derived from a curve that is directly fitted to the shape of the quantile-quantile plot, possibly giving greater weight to observations in this curve fitting process to regions of the distribution that the user is most interested in analysing. His suggested curve form to use for this purpose is a cubic, since the fourth moment Cornish-Fisher approach is in effect also characterised by a cubic quantile-quantile plot but not necessarily one giving the most suitable weights to different parts of the distributional form.

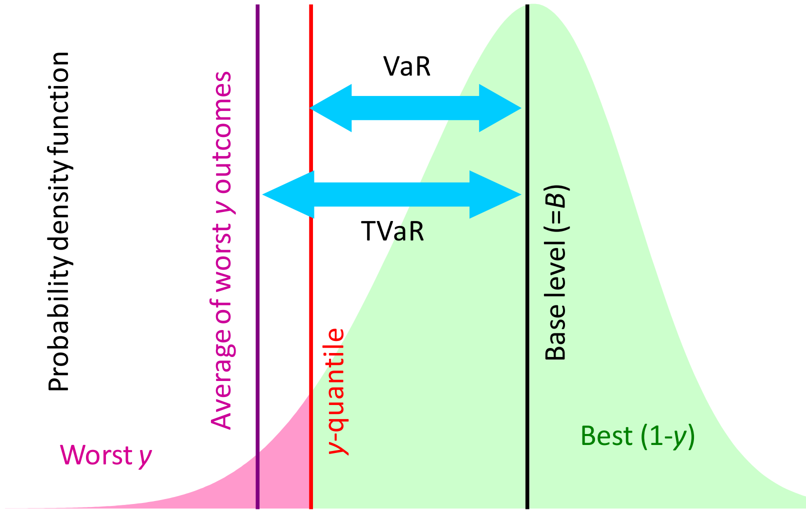
As noted in Kemp (2009) such an approach also simplifies computation of TVaR risk measures.

Suppose the quantile-quantile plot (versus the corresponding standardised normal distribution) takes a cubic form, i.e. is of the form $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Not all choices of $a_0, \ldots, a_3$ correspond to a valid probability distribution. Real vaued probability distributions must have monotonically non-decreasing cumulative distribution functions, which in this instance means that $f'(x) \equiv df/dx$ needs to be non-decreasing for all x, which requires $a_3 \geq 0$ and $a_1 - a_2^2/3a_3 \geq 0$.

If the cubic does correspond to a valid probability distribution then the VaR and TVaR of the distribution, for a given confidence level y, are defined as follows, where $p(x)$ is the distribution's probability density function (assuming that we adopt the same definition for TVaR as is used in the illustrative chart below and suitably rebase the x-axis, i.e. here set $B = 0$):

$$VaR(y) = k \ \ where \ \int_{-\infty}^{-k} p(x)dx = y$$

$$TVaR(y) = -\frac{1}{k} \int_{-\infty}^{-k} xp(x)dx$$



At first sight these integrals look quite complicated to evaluate, since they appear to require us to derive $p(x)$. However we note that in this instance the following relationship applies for an arbitrary $g(x)$ satisfying appropriate regularity conditions, where $q(x)$ is the probability density function of the corresponding normal distribution (with, say, mean μ and standard deviation σ) and $N^{-1}(z)$ is the standard inverse normal function:

$$\int_{-\infty}^{f(z)} g(x)p(x)dx = \int_{-\infty}^{z} g(f(x))q(x)dx$$

VaR corresponds to the case where $g(x) = 1$, i.e. can be evaluated, as we might expect as:

$$N^{-1}\left(\frac{-x-\mu}{\sigma}\right)$$

TVaR corresponds to the case where $g(x) = -x/k$, i.e. $g(f(x)) = -(a_0 + a_1 x + a_2 x^2 + a_3 x^3)/k$ and thus can be derived analytically (as a function of z, and deeming $N^{-1}(z)$ to be 'analytic') using methodologies set out in integrating piecewise polynomials against a Gaussian probability density function.