Special polynomials

A (n'th order) polynomial is a function of the form $y(x) = a_0 + a_1 x + \cdots + a_n x^n$ where the a_i are constant. They appear in many guises in mathematics. Whilst it is common to focus on cases where x is a real number, there are attractions in extending their domain to relate to complex numbers, with the a_i also then being allowed to be complex. For example, a polynomial of order n always then has n (possibly not distinct) roots, i.e. values of x where $y(x) = 0$ but even if all the a_i are real some of these roots may be complex.

Several special types of polynomial have been widely analysed, including:

- Legendre polynomials
- Chebshev polynomials
- Hermite polynomials
- Jacobi polynomials
- Laguerre polynomials

These polynomials all appear in a natural way when we try to approximate a functional form as follows.

Suppose we define a space of real or complex valued (continuous) functions on the interval $[a, b]$. A natural 'scalar product' of two functions $x(t)$ and $y(t)$ is then:

$$S(x, y) = \int_a^b \varphi(t) x(t) \bar{y}(t) dt$$

where $\bar{y}(t)$ is the complex conjugate of $y(t)$ and $\varphi(t)$ is a real continuous non-negative function (with at most finitely many zeros) called the weight function for the given scalar product. If the $x(t)$ and $y(t)$ are limited to real functions then the definition simplifies to the folloing (because the same formulae apply, but the complex

We may then, for example, define $\|f\| = S(f, f)$. If $\|f - g\| = 0$ then f and g are then identical (if continuous) within the interval $[a, b]$. We also have $\|f\| \geq 0$ for all f, so we can view f as a good approximation to g if $\|f - g\|$ is close to zero. Different weight functions then indicate where within the interval $[a, b]$ we most want the approximation to be accurate.

As with any vector space, we can define a basis of orthogonal elements, $f_0, f_1, \ldots$ (which is here infinite dimensional) which in aggregate 'span' the entire vector space, i.e. here the entire range of (continuous) functions defined on $[a, b]$. By orthogonal we mean $S(f_i, f_j) = 0$ for $i \neq j$. The different special functions listed above provide natural orthogonal bases for different weight functions:

Legendre: $\varphi(t) = 1$ and $[a, b] = [-1, 1]$ (can also be viewed as a special case of Jacobi with $\alpha = \beta = 0$)

Jacobi: $\varphi(t) = (1 - t)^\alpha (1 + t)^\beta$ and $[a, b] = [-1, 1]$

Chebyshev: the special case of the Jacobi with $\alpha = \beta = -1/2$ which means that they can be expressed in a simple analytical manner.

Laguerre: $\varphi(t) = e^{-t}$ and $[a, b] = [0, \infty]$

Hermite: $\varphi(t) = e^{-t^2}$ and $[a, b] = [-\infty, \infty]$

For example, the first few Legendre polynomials are $P_0(t) = 0$, $P_1(t) = t$, $P_2(t) = \frac{1}{2}(3t^2 - 1)$, $P_3(t) = \frac{1}{2}(5t^3 - 3t)$, …

The exact definition of each special polynomial type depends on the 'normalisation' used. This is because if $S(f_i, f_j) = 0$ then $S(kf_i, f_j) = 0$ for any k. The usual normalisation convention involves $\|f_i\| = 1$ for all i.

In the financial world, the computation of many types of risk measures is mathematically akin to a evaluating a particular integral. A common way of carrying out numerical integration is to use an approach called Gaussian quadrature. This is often implemented in a fashion that makes use of some of the polynomials described above.