Probability distributions

[Nematrian website page: ProbabilityDistributionsIntro, © Nematrian 2020]

The Nematrian website contains information and analytics on a wide range of probability distributions, including:

Discrete (univariate) distributions

- Bernoulli, see also binomial distribution
- Binomial
- Geometric, see also negative binomial distribution
- Hypergeometric
- Logarithmic
- Negative binomial
- Poisson
- Uniform (discrete)

Continuous (univariate) distributions

- Beta
- Beta prime
- Burr
- Cauchy
- Chi-squared
- Dagum
- Degenerate
- Error function
- Exponential
- *F*
- Fatigue, also known as the Birnbaum-Saunders distribution
- Fréchet, see also generalised extreme value (GEV) distribution
- Gamma
- Generalised extreme value (GEV)
- Generalised gamma
- Generalised inverse Gaussian
- Generalised Pareto (GDP)
- Gumbel, see also generalised extreme value (GEV) distribution
- Hyperbolic secant
- Inverse gamma
- Inverse Gaussian
- Johnson SU
- Kumaraswamy
- Laplace
- Lévy
- Logistic
- Log-logistic
- Lognormal
- Nakagami
- Non-central chi-squared
- Non-central *t*

- Normal
- Pareto
- Power function
- Rayleigh
- Reciprocal
- Rice
- Student's *t*
- Triangular
- Uniform
- Weibull, see also generalised extreme value (GEV) distribution

Continuous multivariate distributions

- Inverse Wishart

Copulas (a copula is a special type of continuous multivariate distribution)

- Clayton
- Comonotonicity
- Countermonotonicity (only valid for $n = 2$, where n is the dimension of the input)
- Frank
- Generalised Clayton
- Gumbel
- Gaussian
- Independence
- *t*

The Nematrian website functions for fitting univariate distributions, creating random variates and calculating moments etc. now cover most of the above probability distributions, see ProbabilityDistributionsFunctions. In many cases these functions can cater for the traditional (textbook) forms of these distributions and variants that include additional shift and scale parameters.

location (i.e. shift) and scale variants

[ProbabilityDistributionsIntro2]

The location and scale of any probability distribution can be adjusted by using the (linear) transform $Y = g + hX$ where g and h are constants (g adjusts the location, i.e. shifts the distribution, whilst h adjusts its scale). This leaves the skew and (excess kurtosis) unaltered but alters the mean and variance as $E(Y) = g + hE(X)$ and $var(Y) = h^2 var(X)$.

In some cases the typical distributional specification already includes such components. For example, the normal distribution $N(\mu, \sigma^2)$ is the location and scale adjusted version of the unit normal distribution $N(0,1)$.

In other cases the standard distributional specification does not include such adjustments. For example, the (standard) Student's t distribution depends one just one parameter, its degrees of freedom. The probability distribution orientated Nematrian web functions recognise location and/or scale adjusted variants of wide range of standard probability distributions.

Discrete (univariate) distributions

The Bernoulli distribution

[BernoulliDistribution]

A Bernoulli trial is an experiment that has one of two possible outcomes, 'success' with probability p and 'failure' with probability $1-p$. The Bernoulli distribution is a probability distribution that takes the value of 1 if such a trial is a 'success' and 0 if it is a 'failure'.

The Bernoulli distribution is a special case of the binomial distribution, $B(n,p)$, with $n=1$. For characteristics of the Bernoulli distribution (e.g. mean, standard deviation etc.), please refer to the corresponding characteristics for the binomial distribution.

For other probability distributions see here.

The binomial distribution

[BinomialDistribution]

The binomial distribution $B(n,p)$ is the discrete probability distribution applicable to the number of successes in a sequence of n independent yes/no experiments each of which has a success probability of p. Each individual success/failure experiment is called a Bernoulli trial, so if $n=1$ then the binomial distribution is a Bernoulli distribution.

It has the following characteristics:

Distribution name	**Binomial distribution**
Common notation	$X \sim B(n,p)$
Parameters	n = number of (independent) trials, positive integer p = probability of success in each trial, $0 \le p \le 1$
Support	$x \in \{0,1,\ldots,n\} = number\ of\ successes$
Probability mass function	$f(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$
Cumulative distribution function	$F(x) = \sum_{j=0}^{x} \binom{n}{j} p^j (1-p)^{n-j} = I_{1-p}(n-x, x+1)$
Mean	np
Variance	$np(1-p)$
Skewness	$\frac{1-2p}{\sqrt{np(1-p)}}$
(Excess) kurtosis	$\frac{1-6p(1-p)}{np(1-p)}$
Characteristic function	$(1-p+pe^{it})^n$
Other comments	The Bernoulli distribution is $B(1,p)$ and corresponds to the likelihood of success of a single experiment. Its probability mass function and cumulative distribution function are: $f(x) = F(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \end{cases}$

	The Bernoulli distribution with $p = 1/2$, i.e. $B(1, 1/2)$, has the minimum possible excess kurtosis, i.e. -2. The mode of $B(n, p)$ is $int\big((n+1)p\big)$ if $(n+1)p$ is 0 or not an integer and is n if $(n+1)p = n+1$. If $(n+1)p \in \{1,2,\dots,n\}$ then the distribution is bi-modal, with modes $(n+1)p$ and $(n+1)p-1$.

The binomial distribution is often used to model the number of successes in a sample size of n from a population size of N. Since such samples are not independent, the resulting distribution is actually a hypergeometric distribution and not a binomial distribution. However if $N \gg n$ then the binomial distribution becomes a good approximation to the relevant hypergeometric distribution and is thus often used.

In the above $\binom{n}{x}$ is the binomial coefficient.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "binomial". For details of other supported probability distributions see here.

The geometric distribution

[GeometricDistribution]

The geometric distribution describes the probability of x successes in a sequence of independent experiments each with likelihood of success of p that arise before there is 1 failure. It is a special case of the negative binomial distribution.

Note: different texts adopt slightly different definitions, e.g. it may be the total number of trials (i.e. 1 more than the above) in which case it may be called the *shifted* geometric distribution and/or p may denote the probability of failure rather than the probability of success.

The hypergeometric distribution

[HypergeometricDistribution]

The hypergeometric distribution describes the probability of x successes in n draws from a finite population size N containing m successes *without* replacement. This contrasts with the binomial distribution which describes the probability of x successes in n draws *with* replacement

Distribution name	**Hypergeometric distribution**
Common notation	$X \sim Hypergeometric(m, N, n)$
Parameters	N = population size, integral ($N > 0$) n = sample size, integral ($0 < n \leq N$) m = number of tagged items, integral ($0 < m \leq N$)
Domain	$\max(0, n+m-N) \leq x \leq \max(n, m)$, x an integer
Probability mass function	$f(x) = \dfrac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$

Cumulative distribution function	$F(x) = (x) = \sum_{j=0}^{x} \frac{\binom{m}{j}\binom{N-m}{n-j}}{\binom{N}{j}} = 1 - \frac{\binom{n}{k+1}\binom{N-n}{m-k-1}}{\binom{N}{m}} Y$ where $Y = {}_3F_2(1, x+1-m, x+1-n; x+2, N+x+2-m-n; 1)$ ${}_pF_q$ is the generalised hypergeometric function, i.e. ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}$ and $(a)_n$ involves the rising factorial or Pochhammer notation, i.e. $(a)_n = a(a+1)(a+2) \dots (a+n-1)$ and $(a)_0 = 1$
Mean	$\frac{nm}{N}$
Variance	$\frac{nm(N-m)(N-n)}{N^2(N-1)}$
Skewness	$\frac{(N-2m)(N-1)^{1/2}(N-2n)}{\left(nm(N-m)(N-n)\right)^{1/2}(N-2)}$
(Excess) kurtosis	$\frac{A+B}{C}$ where $A = (N-1)N^2\left(N(N+1) - 6m(N-m) - 6n(N-m)\right)$ $B = 6nm(N-m)(N-n)(5N-6)$ $C = nm(N-m)(N-n)(N-2)(N-3)$
Characteristic function	$\frac{\binom{N-m}{n}}{\binom{N}{n}} {}_2F_1(-n, -m; n-m-n+1; e^{it})$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "hypergeometric". For details of other supported probability distributions see here.

The logarithmic distribution

[LogarithmicDistribution]

The logarithmic distribution arises from following power series expansion:

$$-\log(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \cdots$$

This means that the function $f(x) = -\frac{p^x}{x\log(1-p)}$, $x = 1,2,3,\dots$ can naturally be interpreted as a probability mass function since $\sum_{k=1}^{\infty} f(k) = 1$.

Distribution name	**Logarithmic distribution**
Common notation	$X \sim Log(p)$
Parameters	p = shape parameter ($0 < p < 1$)
Domain	$1 \le x < +\infty, x$ an integer

Probability mass function	$$f(x) = -\frac{p^x}{x\log(1-p)}$$
Cumulative distribution function	$$F(x) = -\frac{1}{\log(1-p)}\sum_{j=1}^{x}\frac{p^j}{j} = 1 + \frac{B_p(x+1,0)}{\log(1-p)}$$
Mean	$$-\frac{p}{(1-p)\log(1-p)}$$
Variance	$$-\frac{p(p+\log(1-p))}{(1-p)^2(\log(1-p))^2} = V$$
Skewness	$$-\frac{p}{(1-p)^3V^{3/2}\log(1-p)}\left(1+p+\frac{3p}{\log(1-p)}+\frac{2p^2}{\log^2(1-p)}\right)$$
(Excess) kurtosis	$$-\frac{p}{(1-p)^4V^2\log(1-p)}A-3$$ where $$A=\left(1+4p+p^2+\frac{4p(1+p)}{\log(1-p)}+\frac{6p^2}{\log^2(1-p)}+\frac{3p^3}{\log^3(1-p)}\right)$$
Characteristic function	$$\frac{\log(1-pe^{it})}{\log(1-p)}$$
Other comments	The logarithmic distribution has a mode of 1. If N is a random variable with Poission distribution and X_i, $i=1,\ldots$ is an infinite sequence of iid random variables each distributed $Log(p)$ then $Y=\sum_{i=1}^{N}X_i$ has a negative binomial distribution showing that the negative binomial distribution is an example of a compound Poisson distribution

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "logarithmic". For details of other supported probability distributions see here.

The negative binomial distribution

[NegativeBinomialDistribution]

The negative binomial distribution describes the probability of x successes in a sequence of independent experiments each with likelihood of success of p that arise before there are r failures. In this interpretation r is a positive integer, but the distributional definition can also be extended to real values of $r>0$. Note: different texts adopt slightly different definitions, e.g. with support starting at $x=r$ not $x=0$ and/or with p denoting probability of failure rather than probability of success.

Distribution name	**Negative binomial distribution**
Common notation	$X \sim NB(k,p)$
Parameters	r = number of failures ($r>0$) p = probability of success in each experiment ($0<p<1$)
Support	$x \in \{0,1,2,\ldots\}$
Probability mass function	$$f(x) = \binom{x+r-1}{x}p^x(1-p)^r$$ If r is non-integral then is:

	$$f(x)=\frac{\Gamma(x+r)}{\Gamma(r)\Gamma(x+1)}p^x(1-p)^r$$
Cumulative distribution function	$F(x)=1-I_p(x+1,r)$
Mean	$\frac{pr}{1-p}$
Variance	$\frac{pr}{(1-p)^2}$
Skewness	$\frac{1+p}{\sqrt{pr}}$
(Excess) kurtosis	$\frac{6}{r}+\frac{(1-p)^2}{pr}$
Characteristic function	$\left(\frac{1-p}{1-pe^{it}}\right)^r$
Other comments	The *geometric* distribution is the same as the negative binomial distribution with parameter $r=1$. Its pdf and cdf are therefore: $$f(x)=p(1-p)^x$$ $$F(x)=1-(1-p)^{x+1}$$ For the special case where r is an integer the negative binomial distribution is also called the *Pascal* distribution. The Poisson distribution is also a limiting case of the negative binomial: $$Poisson(\lambda)=\lim_{r\to\infty}NB\left(r,\frac{r}{\lambda+r}\right)$$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "negative binomial". For details of other supported probability distributions see here.

The Poisson distribution

[PoissonDistribution]

The Poisson distribution expresses the probability of a given number of events occurring in a fixed interval of time if the events occur with a known average rate and independently of the time since the last event.

Distribution name	**Poisson distribution**
Common notation	$X\sim Pois(\lambda)$
Parameters	λ = event rate ($\lambda>0$)
Support	$x\in\{0,1,2,\dots\}$
Probability mass function	$$f(x)=\frac{\lambda^x e^{-\lambda}}{x!}$$
Cumulative distribution function	$$F(x)=e^{-\lambda}\sum_{j=0}^{x}\frac{\lambda^j}{j!}$$ (can also be expressed using the incomplete gamma function)
Mean	λ
Variance	λ

Skewness	$\lambda^{-1/2}$
(Excess) kurtosis	λ^{-1}
Characteristic function	$e^{\lambda(e^{it}-1)}$
Other comments	The median is approximately $int(\lambda + 1/3 - 0.02/\lambda)$. The mode is $int(\lambda)$ if λ is not integral. Otherwise the distribution is bi-modal with modes λ and $\lambda - 1$.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "poisson". For details of other supported probability distributions see here.

The uniform (discrete) distribution

[UniformDiscreteDistribution]

The uniform (discrete) distribution involves equally probable outcomes that are spaced uniform intervals apart.

Distribution name	**Uniform (discrete) distribution**
Common notation	$X \sim U(a,b,h)$
Parameters	a = lower limit b = upper limit ($a < b$) h = step size ($b - a = (n-1)h$ where n is a positive integer)
Support	$x \in \{a, a+h, \dots, a+(n-1)h(=b)\}$
Probability mass function	$f(x) = \frac{1}{n} \quad for \;\; x = a, a+h, \dots, a+(n-1)h$
Cumulative distribution function	$F(x) = \begin{cases} 0 & for\ x < a \\ int\left(\frac{x-a}{h} + \frac{1}{n}\right) & for\ a \le x < b \\ 1 & for\ x \ge b \end{cases}$
Mean	$\frac{a+b}{2}$
Variance	$\frac{(b-a)(b-a+2h)}{12}$
Skewness	0
(Excess) kurtosis	$-\frac{6(n^2+1)}{5(n^2-1)}$
Characteristic function	$\frac{1}{n}\left(\frac{e^{iat} - e^{i(b+1)t}}{1 - e^{it}}\right)$
Other comments	The median of this distribution is the same as its mean.

Nematrian web functions

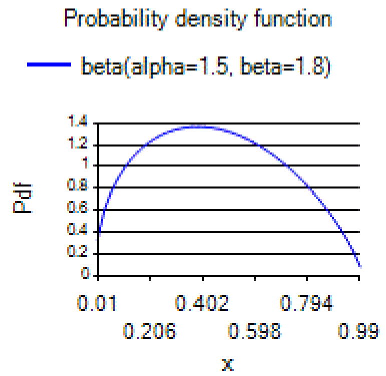
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "uniform (discrete)". For details of other supported probability distributions see here.

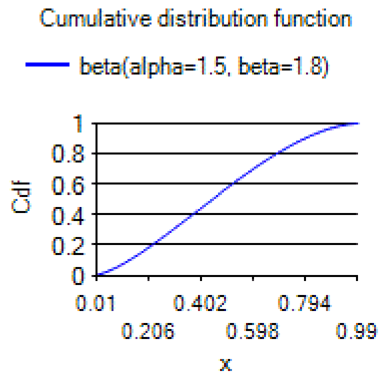
Continuous (univariate) distributions

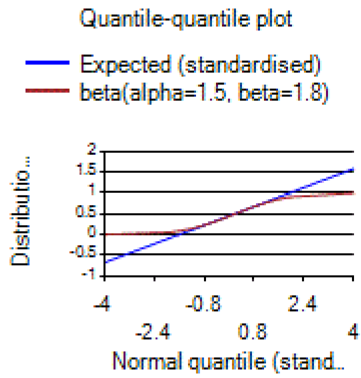
The Beta distribution

[BetaDistribution]

The beta distribution describes a distribution in which outcomes are limited to a specific range, the probability density function within this range being characterised by two shape parameters.







Distribution name	**Beta distribution**
Common notation	$X \sim Beta(\alpha,\beta)$
Parameters	α = shape parameter ($\alpha > 0$) β = shape parameter ($\beta > 0$)
Domain	$0 \le x \le 1$
Probability density function	$f(x) = \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$
Cumulative distribution function	$F(x) = \dfrac{B_x(\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$
Mean	$\dfrac{\alpha}{\alpha+\beta}$
Variance	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Skewness	$\dfrac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{(\alpha+\beta+2)\sqrt{\alpha\beta}}$
(Excess) kurtosis	$\dfrac{6\big((\alpha-\beta)^2(\alpha+\beta+1)-\alpha\beta(\alpha+\beta+2)\big)}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)}$
Characteristic function	${}_1F_1\ (\alpha;\alpha+\beta;it) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\dfrac{\alpha+r}{\alpha+\beta+r}\right)\dfrac{(it)^k}{k!}$
Other comments	The beta distribution is also known as a *beta distribution of the first kind*. Its mode is $\frac{\alpha-1}{\alpha+\beta-2}$ for $\alpha > 1, \beta > 1$. There is no simple closed form solution for its median. The beta distribution parameters are sometimes taken to include boundary parameters a, b ($a < b$) in which case its domain is $a \le x \le b$, and its pdf and cdf are $f(x) = \frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{B(\alpha,\beta)}$ and $F(x) =$

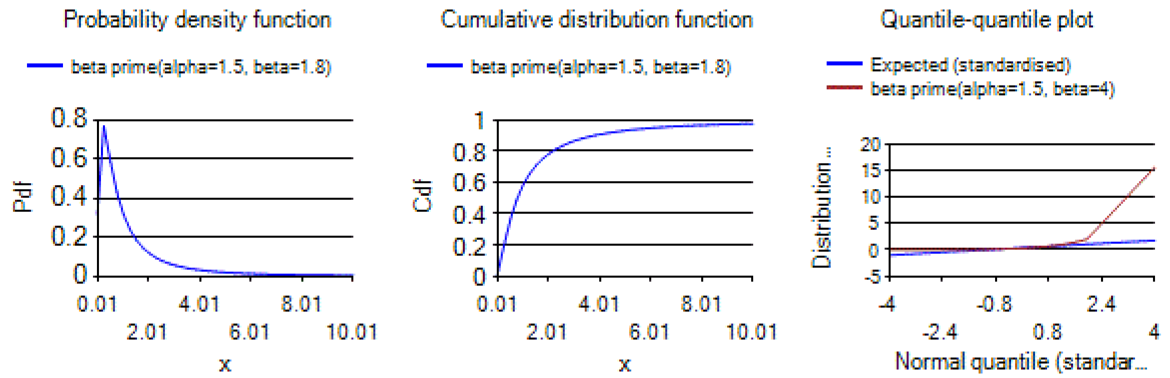
	$\frac{B_z(\alpha,\beta)}{B(\alpha,\beta)}$ where $z=\frac{x-a}{b-a}$, its mean is $\frac{\alpha b+\beta a}{\alpha+\beta}$, its mode is $\frac{(\alpha-1)b+(\beta-1)a}{\alpha+\beta-2}$ for $\alpha>1,\beta>1$ and its variance is $\frac{\alpha\beta(b-a)^2}{(\alpha+\beta)^2(\alpha+\beta+1)}$ If $X\sim\Gamma(\alpha,\theta)$ and $Y\sim\Gamma(\beta,\theta)$ then $\frac{X}{X+Y}\sim Be(\alpha,\beta)$. If $X\sim Beta(\alpha,\beta)$ then $X/(1-X)\sim BetaPrime(\alpha,\beta)$ and if $X\sim Beta\left(\frac{n}{2},\frac{m}{2}\right)$ then $\frac{mX}{n(1-X)}\sim F(n,m)$ (if $n>0$ and $m>0$). The k'th order statistic of a sample of size n from the uniform distribution has a beta distribution, $u_{(k)}\sim Beta(k,n+1-k)$. If $X\sim Beta(1+\lambda(m-a)/(b-a),1+\lambda(b-m)/(b-a))$ then $a+X(b-a)\sim Pert(a,b,m,\lambda)$, i.e. the Pert distribution is a special case of the beta distribution. Its non-central moments are $E(X^r)=\prod_{j=0}^{k-1}\frac{\alpha+j}{\alpha+\beta+j}$.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "beta". Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of "beta4", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Beta prime distribution

[BetaPrimeDistribution]



Distribution name	**Beta prime distribution**
Common notation	$X\sim\beta'(\alpha,\beta)$
Parameters	α = shape parameter ($\alpha>0$) β = shape parameter ($\beta>0$)
Domain	$x\geq 0$
Probability density function	$f(x)=\frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$
Cumulative distribution	$F(x)=I_{x/(1+x)}(\alpha,\beta)$

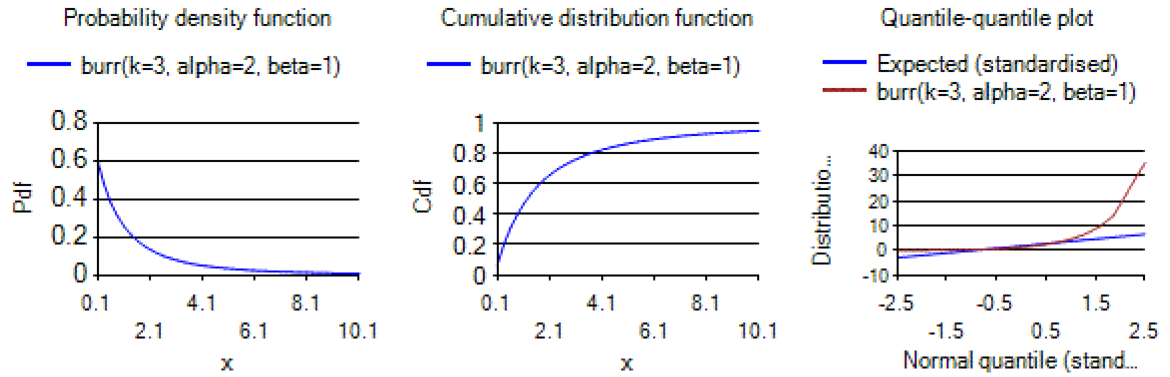
function	
Mean	$\dfrac{\alpha}{\beta-1} \quad if\ \beta > 1$
Variance	$\dfrac{\alpha(\alpha+\beta-1)}{(\beta-1)^2(\beta-2)} \quad if\ \beta > 2$
Other comments	The beta prime distribution is also called the *inverted beta* or the *beta distribution of the second kind* or the *Pearson Type 6 distribution*. The mode of the beta prime distribution is $\frac{\alpha-1}{\alpha+\beta-2}$ for $\alpha > 1, \beta > 1$. There is no simple closed form expression for its median. Its non-central moments (for integral r) are: $$E(X^r) = \prod_{j=1}^{k} \frac{\alpha+j-1}{\beta-i} = \frac{B(\alpha+r)B(\beta-r)}{B(\alpha,\beta)}$$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "beta prime". Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of "beta prime4", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Burr distribution

[BurrDistribution]



Distribution name	**Burr**
Common notation	$X \sim Burr(\alpha, \beta, k)$ or $X \sim SM(\alpha, \beta, k)$
Parameters	k = shape parameter ($k > 0$) α = shape parameter ($\alpha > 0$) β = shape parameter ($\beta > 0$)
Domain	$0 \leq x < +\infty$
Probability density function	$f(x) = \dfrac{\alpha\beta k^\alpha x^{\beta-1}}{(k+x^\beta)^{\alpha+1}}$
Cumulative distribution function	$F(x) = 1 - \left(\dfrac{k}{k+x^\beta}\right)^\alpha$

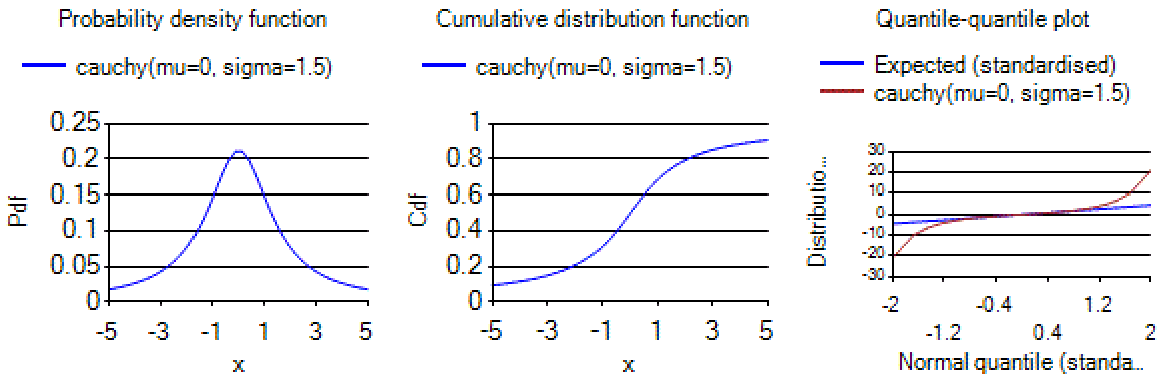
Mean	$\Gamma\left(\alpha-\frac{1}{\beta}\right)\Gamma\left(1+\frac{1}{\beta}\right)\frac{k^{1/\beta}}{\Gamma(\alpha)}$
Variance	$\left(\Gamma\left(\alpha-\frac{2}{\beta}\right)\Gamma\left(1+\frac{2}{\beta}\right)-\frac{\left(\Gamma\left(\alpha-\frac{1}{\beta}\right)\Gamma\left(1+\frac{1}{\beta}\right)\right)^2}{\Gamma(\alpha)}\right)\frac{k^{2/\beta}}{\Gamma(\alpha)}$
Other comments	The Burr distribution is also known as the *Burr Type XII* distribution or the *Singh-Maddala* distribution (sometimes also called the generalised log-logistic distribution). Its non-central moments are: $E(X^r)=\Gamma\left(\alpha-\frac{r}{\beta}\right)\Gamma\left(1+\frac{r}{\beta}\right)\frac{k^{r/\beta}}{\Gamma(\alpha)} \quad r=1,2,\dots,r<\alpha\beta$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "burr". Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of "burr5" ", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Cauchy distribution

[CauchyDistribution]



Distribution name	**Cauchy distribution**
Common notation	$X \sim Cauchy(\mu,\sigma)$
Parameters	μ = location parameter σ = scale parameter ($\sigma>0$)
Domain	$-\infty<x<+\infty$
Probability density function	$f(x)=\left(\pi\sigma\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)\right)^{-1}$
Cumulative distribution function	$F(x)=\frac{1}{\pi}\arctan\left(\frac{x-\mu}{\sigma}\right)+\frac{1}{2}$
Mean	Does not exist
Variance	Does not exist

Skewness	Does not exist
(Excess) kurtosis	Does not exist
Characteristic function	$\exp(\mu i t - \sigma \lvert t \rvert)$
Other comments	The quantile function of the Cauchy distribution is: $$Q(p) = \mu + \sigma \tan\left(\pi\left(p - \frac{1}{2}\right)\right)$$ Its median is thus μ. The Cauchy distribution is a special case of the stable (more precisely the sum stable) distribution family. The special case of the Cauchy distribution when $\mu = 0$ and $\sigma = 1$ is called the standard Cauchy distribution. It coincides with the Student's t distribution with one degree of freedom. It has a probability density function of $f(x; 0,1) = \frac{1}{\pi(1+x^2)}$. If $X \sim N(0,1)$ and $Y \sim N(0,1)$ are independent random variables then $\frac{X}{Y} \sim Cauchy(0,1)$ and this can be used to generate random variates. The Cauchy distribution is also known as the Cauchy-Lorentz or Lorentz distribution (especially amongst physicists).

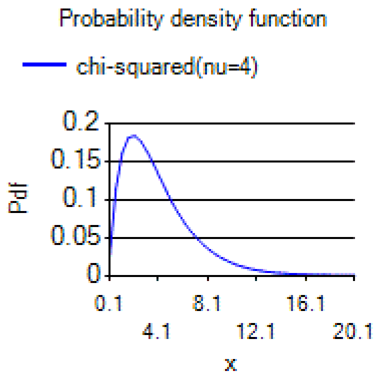
Nematrian web functions

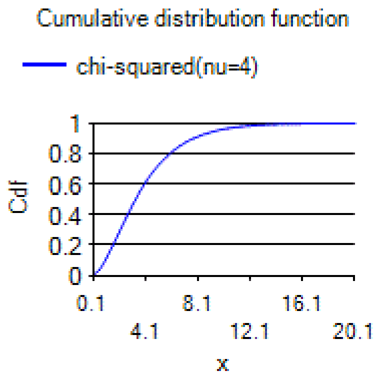
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "cauchy". For details of other supported probability distributions see here.

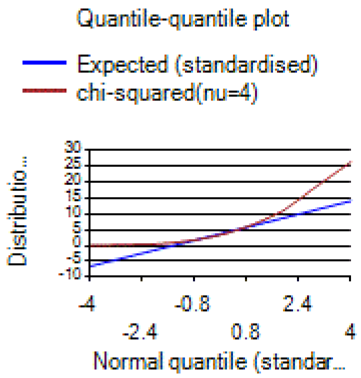
The Chi-squared distribution

[ChiSquaredDistribution]

The chi-squared distribution with ν degrees of freedom is the distribution of a sum of the squares of ν independent standard normal random variables. A consequence is that the sum of independent chi-squared variables is also chi-squared distributed. It is widely used in hypothesis testing, goodness of fit analysis or in constructing confidence intervals. It is a special case of the gamma distribution.







Distribution name	**Chi-squared distribution**

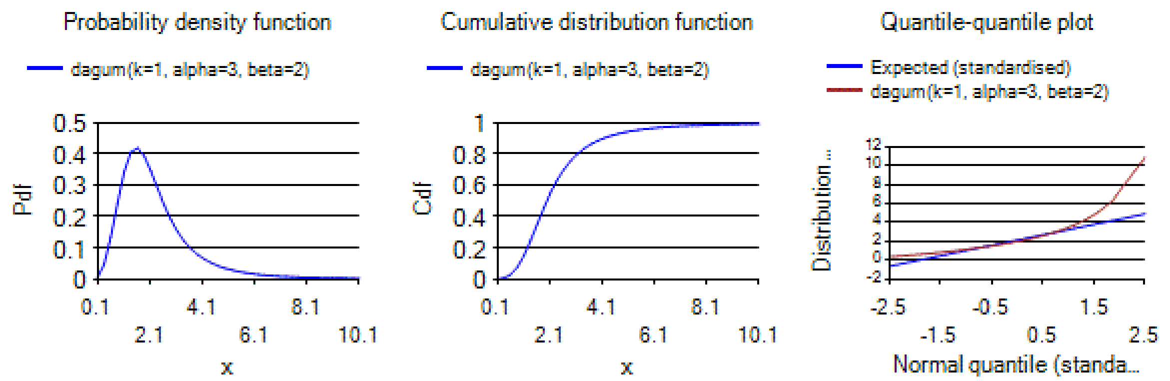
Common notation	$X \sim \chi_\nu^2$
Parameters	ν = degrees of freedom (positive integer)
Domain	$0 \le x < +\infty$
Probability density function	$f(x) = \dfrac{x^{\nu/2-1} \exp\left(-\frac{x}{2}\right)}{2^{\nu/2}\Gamma(\nu/2)}$
Cumulative distribution function	$F(x) = \dfrac{\Gamma_{x/2}(\nu/2)}{\Gamma(\nu/2)}$
Mean	ν
Variance	2ν
Skewness	$2\sqrt{\dfrac{2}{\nu}}$
(Excess) kurtosis	$\dfrac{12}{\nu}$
Characteristic function	$(1-2it)^{-\nu/2}$
Other comments	Its median is approximately $\nu\left(1-\frac{2}{9\nu}\right)^3$. Its mode is $\max(\nu-2,0)$. Is also known as the central chi-squared distribution (when there is a need to contrast it with the noncentral chi-squared distribution). In the special case of $\nu = 2$ the cumulative distribution function simplifies to $F(x) = 1 - e^{-x/2}$. As $\nu \to \infty$, $(\chi_\nu^2 - \nu)/\sqrt{2\nu} \to N(0,1)$ and $qF(q,\nu) \to \chi_q^2$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "chi-squared". Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of "chi-squared3" ", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Dagum distribution

[DagumDistribution]



Distribution name	**Dagum distribution**

Common notation	$X \sim Dagum(\alpha, \beta, k)$
Parameters	k = shape parameter ($k > 0$) α = shape parameter ($\alpha > 0$) β = scale parameter ($\beta > 0$)
Domain	$0 \leq x < +\infty$
Probability density function	$$f(x) = \frac{\alpha k \left(\frac{x}{\beta}\right)^{\alpha k - 1}}{\beta \left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{k+1}}$$
Cumulative distribution function	$$F(x) = \frac{x^{\alpha k}}{(\beta^{\alpha} + x^{\alpha})^{k}}$$
Mean	$$\begin{cases} -\frac{\beta}{\alpha} \frac{\Gamma\left(-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha} + k\right)}{\Gamma(k)} & \alpha > 1 \\ \infty & \text{otherwise} \end{cases}$$
Variance	$$\begin{cases} -\left(\frac{\beta}{\alpha}\right)^2 (A + B) & \alpha > 2 \\ \infty & \text{otherwise} \end{cases}$$ where $$A = \frac{2\alpha\Gamma\left(-\frac{2}{\alpha}\right)\Gamma\left(\frac{2}{\alpha} + k\right)}{\Gamma(k)}$$ $$B = \left(\frac{\Gamma\left(-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha} + k\right)}{\Gamma(k)}\right)^2$$
Other comments	Also known as the Dagum type 1 distribution. Is used in modelling income distributions. The cdf of a Dagum type 2 distribution adds a point mass at the origin and then follows a Dagum type 1 distribution over the positive halfline. Its median is $\beta\left(2^{1/k} - 1\right)^{-1/\alpha}$ and its mode is $\beta\left(\frac{\alpha k - 1}{\alpha + 1}\right)^{1/\alpha}$. Its non-central moments are: $$E(X^r) = \Gamma\left(-\frac{r}{\alpha} + 1\right)\Gamma\left(\frac{r}{\alpha} + k\right)\frac{\beta^r}{\Gamma(k)} \qquad r = 1,2,\ldots, r < \alpha$$ If $X \sim Dagum(\alpha, \beta, k)$ then $1/X \sim Burr(\alpha, 1/\beta, k)$.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "dagum". Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of "dagum4" ", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Degenerate distribution

[DegenerateDistribution]

The degenerate distribution characterises a distribution involving a single outcome. It can be viewed as the limiting case of many common distributions in which the scale parameter tends to zero, so the distribution function concentrates onto a single point. It is also called the Dirac delta function.

Distribution name	**Degenerate distribution (can also be viewed as a discrete distribution)**
Common notation	$X \sim \delta_a$
Parameters	a = location parameter
Domain	$x = a$
Probability density function	$f(x) = \lim_{\sigma \to 0} \left(\exp\left(-\frac{1}{2} \left(\frac{x-a}{\sigma} \right)^2 \right) \right)$
Cumulative distribution function	$f(x) = \begin{cases} 1, & x > a \\ 0, & x < 0 \end{cases}$
Mean	a
Variance	0
Skewness	Does not exist
(Excess) kurtosis	Does not exist
Characteristic function	e^{ita}

Nematrian web functions

Given its degenerate form, no functions relating to the above distribution are accessible via the Nematrian web function library. For details of other supported probability distributions see here.

The error function distribution

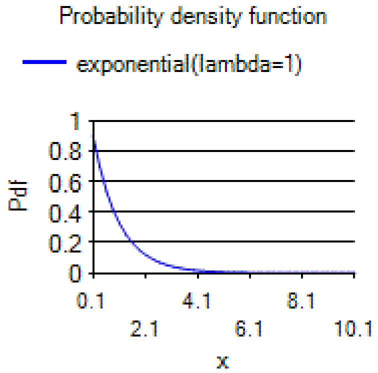
[ErrorFunctionDistribution]

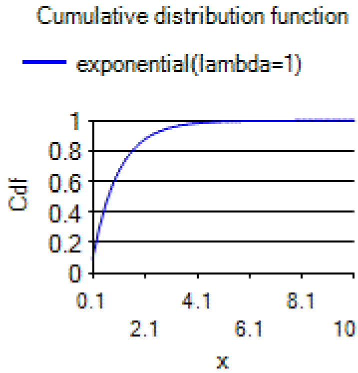
The error function distribution with parameter h is a special case of the normal distribution, i.e. $N\left(0, \frac{1}{2h}\right)$.

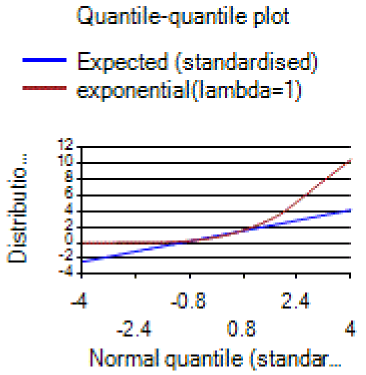
The exponential distribution

[ExponentialDistribution]

The exponential distribution describes the time between events if these events follow a Poisson process (i.e. a stochastic process in which events occur continuously and independently of one another). It is also called the *negative exponential* distribution. It is not the same as the exponential family of distributions.







Distribution name	**Exponential distribution**
Common notation	$X \sim Exp(\lambda)$
Parameters	λ = inverse scale (i.e. rate) parameter ($\lambda > 0$)
Domain	$0 \leq x < +\infty$
Probability density function	$f(x) = \lambda \exp(-\lambda x)$
Cumulative distribution function	$F(x) = 1 - \exp(-\lambda x)$
Mean	$\frac{1}{\lambda}$
Variance	$\frac{1}{\lambda^2}$
Skewness	2
(Excess) kurtosis	6
Characteristic function	$(1 - it/\lambda)^{-1}$
Other comments	The exponential distribution is a special case of the Gamma distribution, as if $X \sim Exp(\lambda)$ then $X \sim \Gamma(1, 1/\lambda)$. The mode of an exponential distribution is 0. The quantile function, i.e. the inverse cumulative distribution function, is $F^{-1}(p; \lambda) = -\frac{\log(1-p)}{\lambda}$. The non-central moments ($r = 1,2,3, \ldots$) are $E(X^r) = \frac{\Gamma(1+r)}{\lambda^r}$. Its median is $\frac{\log 2}{\lambda}$.

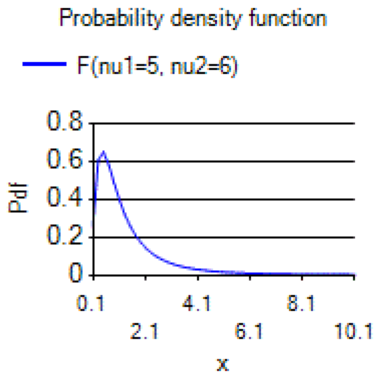
Nematrian web functions

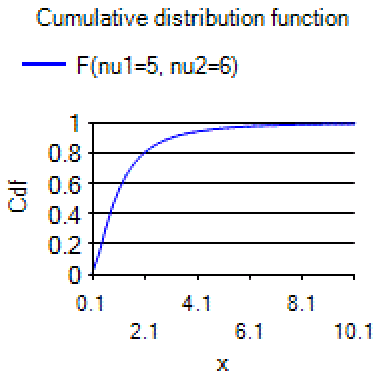
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “exponential”. Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of “exponential2” ”, see also including additional shift and scale parameters. For details of other supported probability distributions see here.

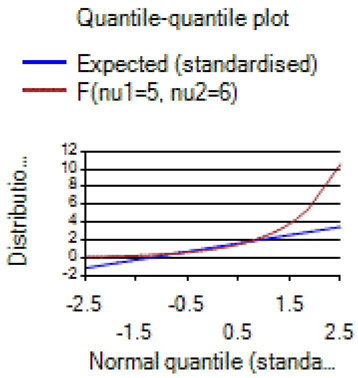
The *F* distribution

[FDistribution]

The *F* distribution is also known as Snedecor's *F* or the Fisher-Snedecor distribution. It commonly arises in statistical tests linked to analysis of variance.







Distribution name	***F* distribution**
Common notation	$X \sim F(\nu_1, \nu_2)$
Parameters	ν_1 = degrees of freedom (first) (positive integer) ν_2 = degrees of freedom (second) (positive integer)
Domain	$0 \le x < +\infty$
Probability density function	$f(x) = \frac{1}{xB(\nu_1/2, \nu_2/2)} \sqrt{\frac{(\nu_1 x)^{\nu_1} \nu_2^{\nu_2}}{(\nu_1 x + \nu_2)^{\nu_1+\nu_2}}}$
Cumulative distribution function	$F(x) = \frac{B_{\nu_1 x/(\nu_1 x + \nu_2)}(\nu_1/2, \nu_2/2)}{B(\nu_1/2, \nu_2/2)} = I_{\nu_1 x/(\nu_1 x+\nu_2)}(\nu_1/2, \nu_2/2)$
Mean	$\frac{\nu_1}{\nu_2 - 2}$ for $\nu_2 > 2$
Variance	$\frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}$ for $\nu_2 > 4$
Skewness	$\frac{(2\nu_1 + \nu_2 - 2)\sqrt{8(\nu_2 - 4)}}{(\nu_2 - 6)\sqrt{\nu_1(\nu_1 + \nu_2 - 2)}}$ for $\nu_2 > 6$
(Excess) kurtosis	$12 \frac{\nu_1(5\nu_2 - 22)(\nu_1 + \nu_2 - 2) + (\nu_2 - 4)(\nu_2 - 2)^2}{\nu_1(\nu_2 - 6)(\nu_2 - 8)(\nu_1 + \nu_2 - 2)}$ for $\nu_2 > 8$
Characteristic function	$\frac{\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)} U\left(\frac{\nu_1}{2}, 1 - \frac{\nu_2}{2}, -\frac{\nu_2}{\nu_1} it\right)$ Where $U(a, b, z)$ is the confluent hypergeometric function of the second kind
Other comments	The *F* distribution is a special case of the Pearson type 6 distribution. It is also a particular example of the beta prime distribution. If $X_1 \sim \chi^2(\nu_1)$ and $X_2 \sim \chi^2(\nu_2)$ are independent random variables then $\frac{X_1/\nu_1}{X_2/\nu_2} \sim F(\nu_1, \nu_2)$ Its mode is $\frac{(\nu_1 - 2)}{\nu_1} \frac{\nu_2}{\nu_2 + 2}$ for $\nu_1 > 2$. There is no simple closed form for the median.

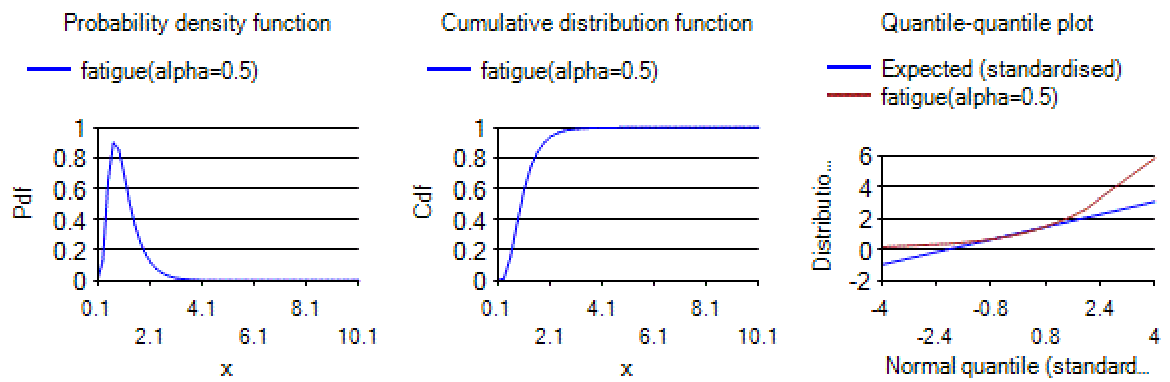
Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "f". Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of "f4", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Fatigue distribution

[FatigueDistribution]

The fatigue distribution, also known as the *Birnbaum-Saunders* distribution or the fatigue life distribution is used extensively to model failure times.



Distribution name	**(standard) Fatigue distribution**
Common notation	$X \sim Birnbaum-Saunders(\alpha)$
Parameters	α = shape parameter ($\alpha > 0$)
Domain	$0 < x < +\infty$
Probability density function	$f(x) = \frac{\sqrt{x}+\sqrt{1/x}}{2\alpha x}\phi\left(\frac{\sqrt{x}-\sqrt{1/x}}{\alpha}\right)$ where $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$
Cumulative distribution function	$F(x) = N\left(\frac{\sqrt{x}-\sqrt{1/x}}{\alpha}\right)$
Mean	$1+\frac{\alpha^2}{2}$
Variance	$\frac{\alpha^2(4+5\alpha^2)}{4}$
Other comments	Its quantile function, i.e. the inverse of its cdf, is: $F^{-1}(p) = \frac{1}{4}\left(\alpha N^{-1}(p)+\sqrt{4+\left(\alpha N^{-1}(p)\right)^2}\right)^2$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “fatigue”. Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of “fatigue3”, see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Frechét distribution

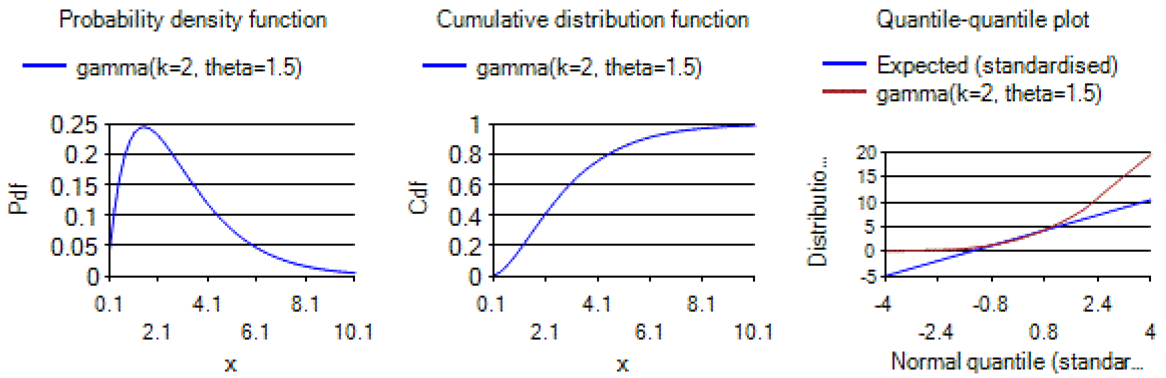
[FrechetDistribution]

The Frechét distribution is a special case of the generalised extreme value (GEV) distribution. It characterises the distribution of ‘block maxima’ under certain (relatively restrictive) conditions in situations where the tail index corresponds to fatter tails than arises with the normal distribution.

The gamma distribution

[GammaDistribution]

The gamma distribution is a two-parameter family of continuous probability distributions. Two different parameterisations are in common use, see below, with the (k,θ) parameterisation being apparently somewhat more common in econometrics and the (k,λ) parameterisation being somewhat more common in Bayesian statistics.



Distribution name	**Gamma distribution**
Common notation	$X\sim\Gamma(k,\theta)$ or $\Gamma(\alpha,\lambda)$
Parameters	Has two commonly used parameterisations: k = shape parameter $(k>0)$ θ = scale parameter $(\theta>0)$ or λ = inverse scale (i.e. rate) parameter $(\theta>0)$ where $\lambda=1/\theta$. Unless otherwise specified the material below assumes the first parameterisation (i.e. using a scale parameter)
Domain	$0\leq x<+\infty$
Probability density function	$f(x)=\dfrac{x^{k-1}e^{-x/\theta}}{\theta^k\Gamma(k)}$
Cumulative distribution function	$F(x)=\dfrac{\Gamma_{x/\theta}(k)}{\Gamma(k)}$
Mean	$k\theta$

Variance	$k\theta^2$
Skewness	$2/\sqrt{k}$
(Excess) kurtosis	$6/k$
Moment generating function	$(1-\theta t)^{-k}$ for $t<1/\theta$
Characteristic function	$(1-i\theta t)^{-k}$
Other comments	The gamma distribution can also be defined with a location parameter, γ, say, in which case its domain is shifted to $\gamma \leq x < +\infty$. Its mode is $(k-1)\theta$ for $k \geq 1$. If X follows an exponential distribution with rate parameter λ then $X \sim \Gamma(1, 1/\lambda)$. If X follows a chi-squared distribution, with υ degrees of freedom, i.e. i.e. $X \sim \chi^2(\upsilon)$ then $X \sim \Gamma(\upsilon/2, 2)$ and $cX \sim \Gamma(\upsilon/2, 2c)$. If k is integral then $\Gamma(k,\theta)$ is also called the *Erlang* distribution. It is the distribution of the sum of k independent exponential variables each with mean $\theta = 1/\lambda$. Events that occur independently with some average rate are commonly modelled using a Poisson process. The waiting times between k occurrences of the event are then Erlang distributed whilst the number of events in a given amount of time is Poisson distributed. If X follows a *Maxwell-Boltzmann* distribution with parameter a then $X^2 \sim \Gamma(3/2, \theta = 2a^2)$. If X follows a skew logistic distribution with parameter θ then $\log(1+e^{-X}) \sim \Gamma(1,\theta)$. The gamma distribution is the conjugate prior for the precision (i.e. inverse variance) of a normal distribution and for the exponential distribution. The gamma distribution has the 'summation' property that if $X_i \sim \Gamma(k_i, \theta)$ for $i = 1, \dots, n$ and the X_i are independent then $\sum_{i=1}^{n} X_i \sim \Gamma(\sum_{i=1}^{n} k_i, \theta)$. Its non-central moments $(r = 1,2,3, \dots)$ are $E(X^r) = \frac{\Gamma(k+r)}{\Gamma(k)}\theta^r$. There is in general no simple closed form for its median.

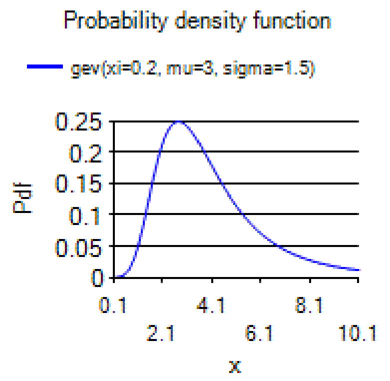
Nematrian web functions

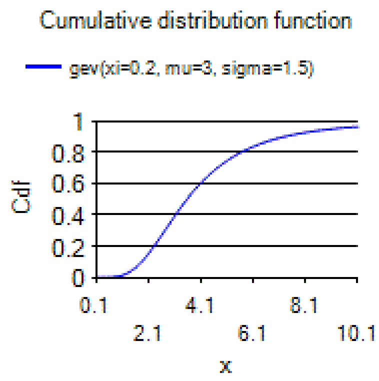
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "gamma". Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of "gamma3", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

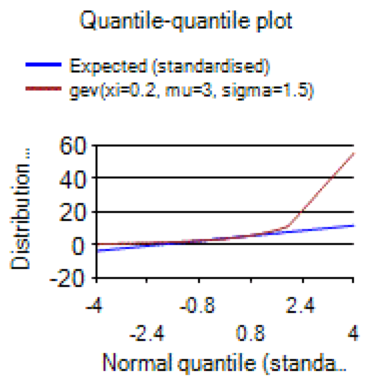
The generalised extreme value distribution

[GEVDistribution]

The generalised extreme value (or generalized extreme value) distribution characterises the behaviour of 'block maxima' under certain (somewhat restrictive) regularity conditions. See also Nematrian's webpages about Extreme Value Theory (EVT).







Distribution name	**Generalised extreme value (GEV) distribution (for maxima)**
Common notation	$X \sim GEV(\xi, \mu, \sigma)$
Parameters	ξ = shape parameter μ = location parameter σ = scale parameter
Domain	$1 + \left(\frac{x-\mu}{\sigma}\right)\xi > 0 \quad \xi \neq 0$ $-\infty < x < \infty \qquad \xi = 0$
Probability density function	$f(x) = \frac{1}{\sigma} Q(x)^{\xi+1} e^{-Q(x)}$ where $Q(x) = \begin{cases} \left(1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\xi} & \xi \neq 0 \\ \exp\left(-\frac{x-\mu}{\sigma}\right) & \xi = 0 \end{cases}$
Cumulative distribution function	$F(x) = e^{-Q(x)}$
Mean	$\begin{cases} \mu + \sigma \frac{\Gamma(1-\xi) - 1}{\xi} & if\ \xi \neq 0, \xi < 1 \\ \mu + \sigma\gamma & if\ \xi = 0 \\ \infty & \xi \geq 1 \end{cases}$ where γ is Euler's constant, i.e. $\lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right)$
Variance	$\begin{cases} \sigma^2 \frac{g_2 - g_1^2}{\xi^2} & if\ \xi \neq 0, \xi < 1/2 \\ \frac{\sigma^2 \pi^2}{6} & if\ \xi = 0 \\ \infty & \xi \geq 1/2 \end{cases}$ Where $g_k = \Gamma(1 - k\xi)$

Skewness	$\begin{cases} \dfrac{g_3 - 3g_1g_2 + 2g_1^3}{(g_2 - g_1^2)^{3/2}} & if\ \xi \neq 0 \\ \dfrac{12\sqrt{6}\varsigma(3)}{\pi^3} & if\ \xi = 0 \end{cases}$ where $\varsigma(x)$ is the Riemann zeta function, i.e. $\sum_{k=1}^{\infty}\frac{1}{k^x}$.
(Excess) kurtosis	$\begin{cases} \dfrac{g_4 - 4g_1g_3 + 6g_2g_1^2 - 3g_1^4}{(g_2 - g_1^2)^2} & if\ \xi \neq 0 \\ \dfrac{12}{5} & if\ \xi = 0 \end{cases}$
Other comments	ξ defines the tail behaviour of the distribution. The sub-families defined by $\xi = 0$ (Type I), $\xi > 0$ (Type II) and $\xi < 0$ (Type III) correspond to the Gumbel, Frechét and Weibull families respectively. An important special case when analysing threshold exceedances involves $\mu = 0$ (and normally $\xi > 0$) and this special case may be referred to as $GEV(\xi, \sigma)$.

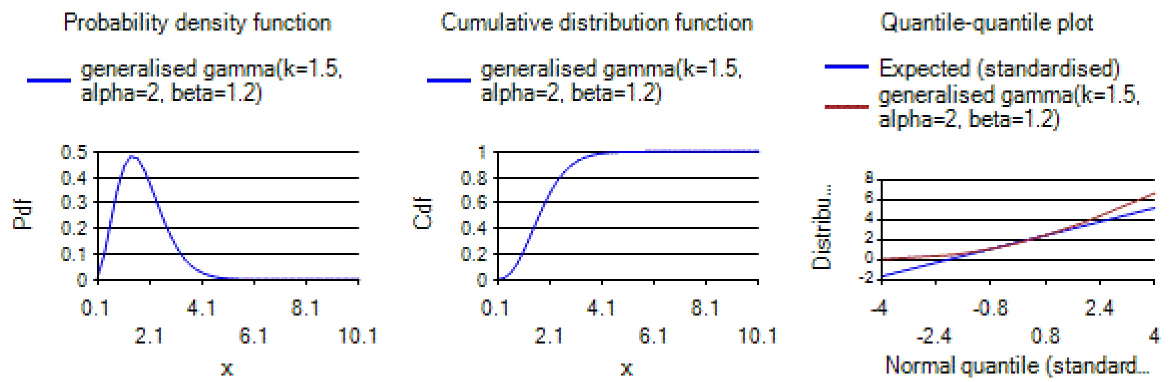
Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "gev". For details of other supported probability distributions see here.

The generalised gamma distribution

[GeneralisedGammaDistribution]

The generalised gamma (or generalized gamma) distribution is a generalisation of the gamma distribution that includes several of the parametric distributions typically used in survival analysis.



Distribution name	**Generalised gamma distribution**
Common notation	$X \sim GeneralisedGamma(k, \alpha, \beta)$
Parameters	k = shape parameter ($k > 0$) α = shape parameter ($\alpha > 0$) β = scale parameter ($\beta > 0$)
Domain	$0 \leq x < +\infty$

Probability density function	$f(x) = \frac{k}{\beta\Gamma(\alpha)}\left(\frac{x}{\beta}\right)^{k\alpha-1}\exp\left(-\left(\frac{x}{\beta}\right)^k\right)$
Cumulative distribution function	$F(x) = \frac{\Gamma_{(x/\beta)^k}(\alpha)}{\Gamma(\alpha)}$
Mean	$\beta\frac{\Gamma\left(\alpha+\frac{1}{k}\right)}{\Gamma(\alpha)}$
Variance	$\beta^2\frac{\Gamma\left(\alpha+\frac{2}{k}\right)\Gamma(\alpha)-\Gamma\left(\alpha+\frac{1}{k}\right)^2}{\Gamma(\alpha)^2}$
Skewness	$\frac{\Gamma\left(\alpha+\frac{3}{k}\right)\Gamma(\alpha)^2-3\Gamma\left(\alpha+\frac{2}{k}\right)\Gamma\left(\alpha+\frac{1}{k}\right)\Gamma(\alpha)+2\Gamma\left(\alpha+\frac{1}{k}\right)^3}{\left(\Gamma\left(\alpha+\frac{2}{k}\right)\Gamma(\alpha)-\Gamma\left(\alpha+\frac{1}{k}\right)^2\right)^{3/2}}$
(Excess) kurtosis	$\frac{A+B+C+D}{\left(\Gamma\left(\alpha+\frac{2}{k}\right)\Gamma(\alpha)-\Gamma\left(\alpha+\frac{1}{k}\right)^2\right)^2}-3$ where: $A = \Gamma\left(\alpha+\frac{4}{k}\right)\Gamma(\alpha)^3$ $B = -4\Gamma\left(\alpha+\frac{3}{k}\right)\Gamma\left(\alpha+\frac{1}{k}\right)\Gamma(\alpha)^2$ $C = 6\Gamma\left(\alpha+\frac{2}{k}\right)\Gamma\left(\alpha+\frac{1}{k}\right)^2\Gamma(\alpha)$ $D = -3\Gamma\left(\alpha+\frac{1}{k}\right)^4$
Other comments	Its non-central moments are: $E(X^r) = \frac{\beta^r\Gamma\left(\alpha+\frac{r}{k}\right)}{\Gamma(\alpha)}$ The Weibull distribution is a special case with $\alpha = 1$. The gamma distribution is a special case with $k = 1$.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "generalised gamma". Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of "generalised gamma4", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The generalised inverse Gaussian distribution

[GeneralisedInverseGaussianDistribution]

Distribution name	**Generalised inverse Gaussian distribution (GIG)**
Common notation	$X \sim GIG(\lambda, \xi, \psi)$
Parameters	λ = parameter ($\lambda > 0$) ξ = parameter ($\xi \geq 0$) ψ = parameter ($\psi \geq 0$)
Domain	$0 < x < +\infty$

Probability density function	$$f(x)=\frac{\left(\sqrt{\psi/\xi}\right)^{\lambda}}{2K_{\lambda}\left(\sqrt{\xi\psi}\right)}x^{\lambda-1}\exp\left(-\frac{1}{2}\left(\frac{\xi}{x}+\psi x\right)\right)$$ where $K_{\lambda}(x)$ is the modified Bessel function of the third kind with index λ.
Mean	$$\frac{\sqrt{\xi}K_{p+1}\left(\sqrt{\xi\psi}\right)}{\sqrt{\psi}K_{p}\left(\sqrt{\xi\psi}\right)}$$
Variance	$$\left(\frac{\xi}{\psi}\right)\left(\frac{K_{p+2}\left(\sqrt{\xi\psi}\right)}{K_{p}\left(\sqrt{\xi\psi}\right)}-\left(\frac{K_{p+1}\left(\sqrt{\xi\psi}\right)}{K_{p}\left(\sqrt{\xi\psi}\right)}\right)^{2}\right)$$
Characteristic function	$$\left(\frac{\psi}{\psi-2it}\right)^{p/2}\frac{K_{p}\left(\sqrt{\xi(\psi-2it)}\right)}{K_{p}\left(\sqrt{\xi\psi}\right)}$$
Other comments	When $\xi>0$ and $\psi>0$ the non-central moments are: $$E(X^{r})=\left(\frac{\xi}{\psi}\right)^{r/2}\frac{K_{\lambda+r}\left(\sqrt{\xi\psi}\right)}{K_{\lambda}\left(\sqrt{\xi\psi}\right)}$$ Some commentators use GIG to refer to the generalised integer gamma distribution (which is not the same as the generalised inverse Gaussian distribution).

Nematrian web functions

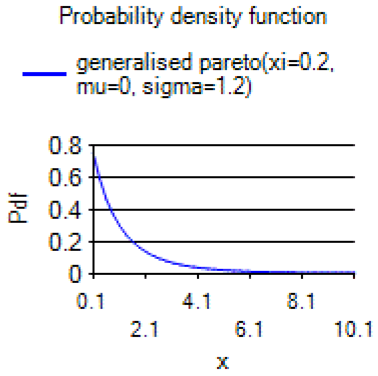
This distribution is not currently supported within the Nematrian web function library. For details of other supported probability distributions see here.

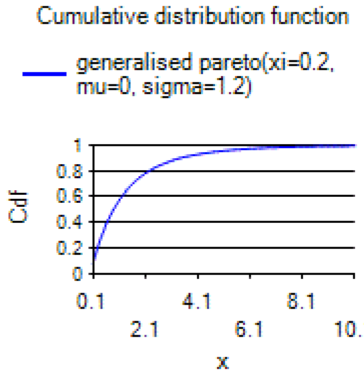
The generalised Pareto distribution

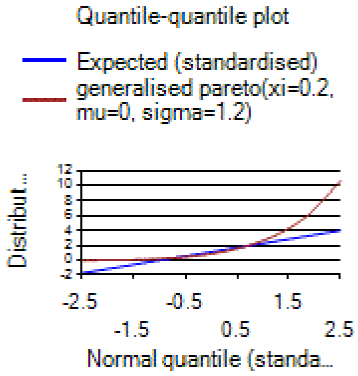
[GeneralisedParetoDistribution]

The generalised Pareto distribution (generalized Pareto distribution) arises in Extreme Value Theory (EVT). If the relevant regularity conditions are satisfied then the tail of a distribution (above some suitably high threshold), i.e. the distribution of 'threshold exceedances', tends to a generalized Pareto distribution.

Care is needed with EVT because what we are in effect doing with it is to extrapolate into the tail of the distribution. Extrapolation is an intrinsically imprecise and subjective mathematical activity. We can in effect view the regularity conditions that need to be satisfied if EVT applies as corresponding to requiring that this extrapolation is done in a particular manner.







Distribution name	**Generalised Pareto distribution (GPD)**
Common notation	$X \sim GPD(\xi, \mu, \sigma)$
Parameters	ξ = shape parameter μ = location parameter σ = scale parameter ($\sigma > 0$)
Domain	$\mu \leq x < +\infty \quad \xi \geq 0$ $\mu \leq x \leq \mu - \frac{\sigma}{\xi} \quad \xi < 0$
Probability density function	$f(x) = \begin{cases} \frac{1}{\sigma}(1+\xi z)^{-1-1/\xi} & \xi \neq 0 \\ \frac{1}{\sigma}\exp(-z) & \xi = 0 \end{cases}$ where $z = \frac{x-\mu}{\sigma}$
Cumulative distribution function	$F(x) = \begin{cases} 1-(1+\xi z)^{-1/\xi} & \xi \neq 0 \\ 1-\exp(-z) & \xi = 0 \end{cases}$
Mean	$\mu + \frac{\sigma}{1-\xi} \qquad \xi < 1$
Variance	$\frac{\sigma^2}{(1-2\xi)(1-\xi)^2} \qquad \xi < \frac{1}{2}$
Skewness	$\frac{2(1+\xi)\sqrt{1-2\xi}}{1-3\xi} \qquad \xi < \frac{1}{3}$
(Excess) kurtosis	$\frac{6(1+\xi-6\xi^2-2\xi^3)}{(1-3\xi)(1-4\xi)} \qquad \xi < \frac{1}{4}$
Other comments	If X is uniformly distributed, $X \sim U(0,1)$ then the variable $Y = \mu + \sigma\left(X^{-\xi}-1\right)/\xi \sim GPD(\mu,\sigma,\xi)$. The mean excess function for a GPD, i.e. $e(u) = E(X-u \mid X > u)$ takes a particularly simple form which is linear in ξ, i.e. $e(u) = \frac{\sigma}{1-\xi} + \frac{\xi(u-\mu)}{1-\xi}$

<u>Nematrian web functions</u>

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "generalised pareto". For details of other supported probability distributions see here.

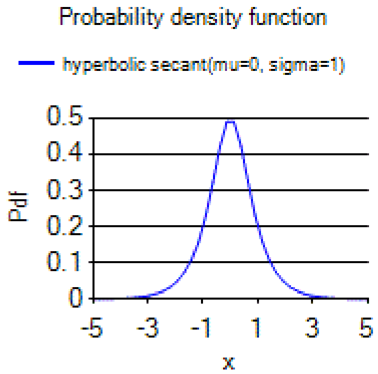
The Gumbel distribution

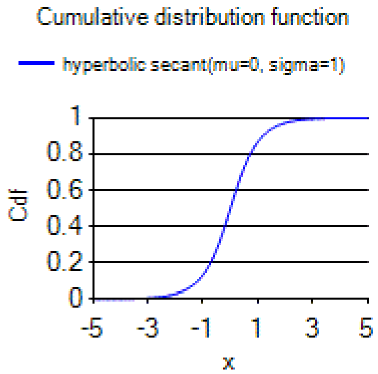
[GumbelDistribution]

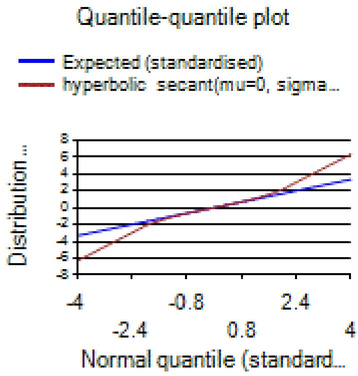
The Gumbel distribution is a special case of the generalised extreme value distribution. It characterises the distribution of 'block maxima' as per Extreme Value Theory (EVT) under certain (relatively restrictive) conditions.

The hyperbolic secant distribution

[Nematrian website page: HyperbolicSecantDistribution, © Nematrian 2015]







Distribution name	**Hyperbolic secant distribution**
Common notation	$X \sim Sech(\mu, \sigma)$
Parameters	σ = scale parameter ($\sigma > 0$) μ = location parameter
Domain	$-\infty < x < +\infty$
Probability density function	$f(x) = \dfrac{\operatorname{sech}\left(\frac{\pi}{2} z\right)}{2\sigma}$ where $z = \dfrac{x-\mu}{\sigma}$
Cumulative distribution function	$F(x) = \dfrac{2}{\pi} \arctan\left(\exp\left(\frac{\pi}{2} z\right)\right)$
Mean	μ
Variance	σ^2
Skewness	0
(Excess) kurtosis	2
Characteristic function	$e^{-it\mu} \operatorname{sech}(\sigma t)$

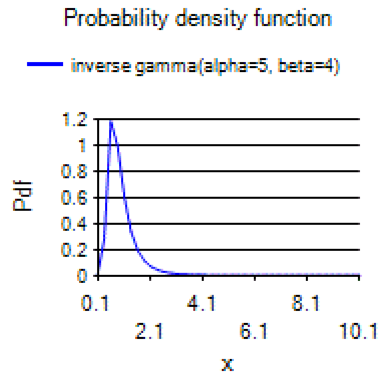
Nematrian web functions

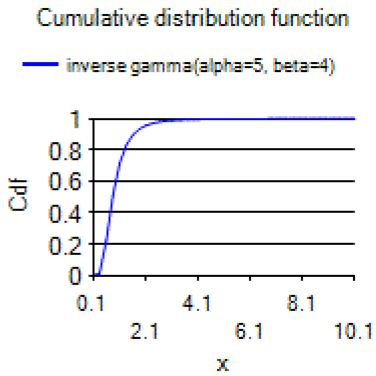
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "hyperbolic secant". For details of other supported probability distributions see here.

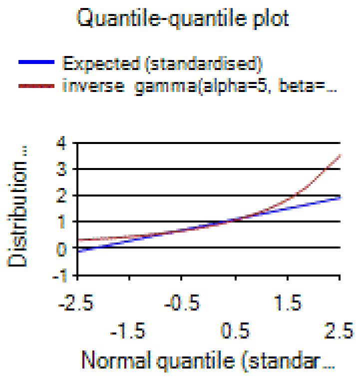
The inverse gamma distribution

[[InverseGammaDistribution]]

The inverse gamma distribution describes the distribution of the reciprocal of a variable distributed according to the gamma distribution.







Distribution name	**Inverse gamma**
Common notation	
Parameters	α = shape parameter ($\alpha > 0$) β = scale parameter ($\beta > 0$)
Domain	$0 < x < +\infty$
Probability density function	$f(x) = \dfrac{\exp\left(-\frac{\beta}{x}\right)}{\beta\Gamma(\alpha)(x/\beta)^{\alpha+1}}$
Cumulative distribution function	$F(x) = 1 - \dfrac{\Gamma_{\beta/x}(\alpha)}{\Gamma(\alpha)}$
Mean	$\dfrac{\beta}{\alpha - 1} \quad for\ \alpha > 1$
Variance	$\dfrac{\beta^2}{(\alpha-1)^2(\alpha-2)} \quad for\ \alpha > 2$
Skewness	$\dfrac{4\sqrt{\alpha-2}}{\alpha-3} \quad for\ \alpha > 3$
(Excess) kurtosis	$\dfrac{30\alpha - 66}{(\alpha-3)(\alpha-4)} \quad for\ \alpha > 4$
Characteristic function	$\dfrac{2(-i\beta t)^{\alpha/2}}{\Gamma(\alpha)} K_\alpha\left(2\sqrt{-i\beta t}\right)$
Other comments	Also called the log Pearson type 5 distribution. Its mode is $\beta/(\alpha+1)$

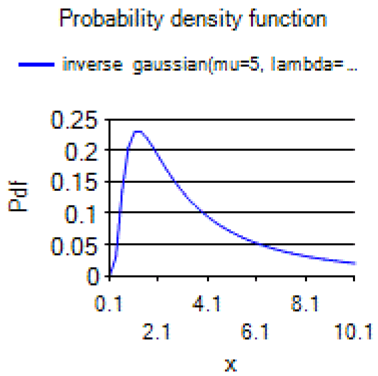
Nematrian web functions

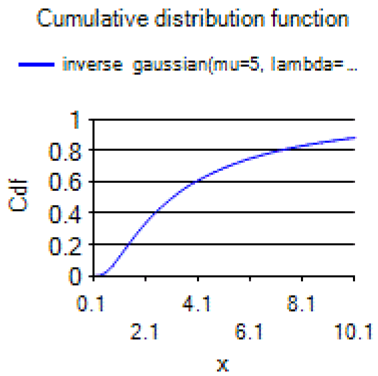
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "inverse gamma". Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of "inverse gamma3", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

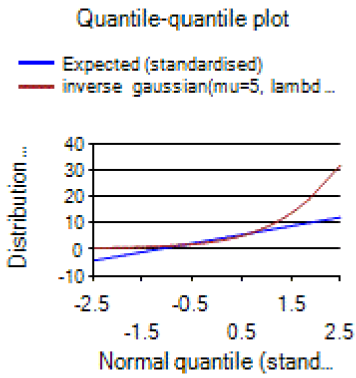
The inverse Gaussian distribution

[InverseGaussianDistribution]

While the Gaussian (i.e. normal distribution describes a Brownian motion's level at a fixed time, the inverse Gaussian distribution describes the distribution of time a Brownian motion starting at 0 with positive drift takes to reach a fixed positive level.







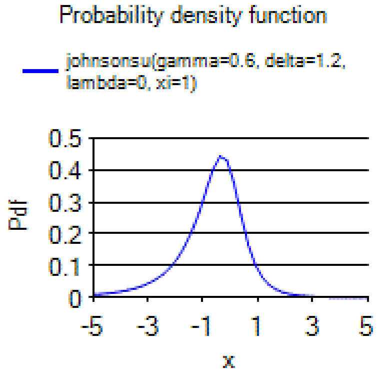
Distribution name	**Inverse Gaussian distribution**
Common notation	$X \sim IG(\lambda, \mu)$
Parameters	λ = parameter ($\lambda > 0$) μ = parameter ($\mu > 0$)
Domain	$0 < x < +\infty$
Probability density function	$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$
Cumulative distribution function	$F(x) = N\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right) + N\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}+1\right)\right)\exp\left(\frac{2\lambda}{\mu}\right)$
Mean	μ
Variance	μ^3/λ
Skewness	$3\left(\frac{\mu}{\lambda}\right)^{1/2}$
(Excess) kurtosis	$\frac{15\mu}{\lambda}$
Characteristic function	$\exp\left(\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2 it}{\lambda}}\right)\right)$

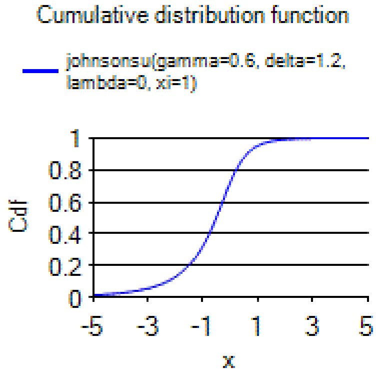
Nematrian web functions

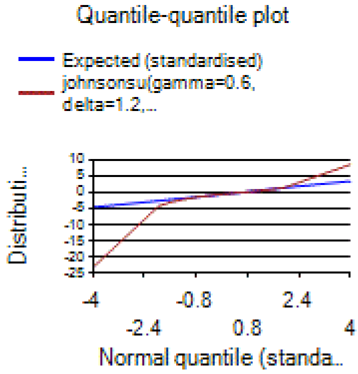
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "inverse gaussian". Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of "inverse gaussian3", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Johnson SU distribution

[JohnsonSUDistribution]







Distribution name	**Johnson SU distribution**
Common notation	$X \sim JohnsonSU(\gamma,\delta,\lambda,\xi)$ or $X \sim S_U(\gamma,\delta,\lambda,\xi)$
Parameters	γ = shape parameter δ = shape parameter ($\delta > 0$) λ = location parameter ξ = scale parameter ($\xi > 0$)
Domain	$-\infty < x < +\infty$
Probability density function	$$f(x) = \frac{\delta \exp\left(-\frac{1}{2}(\gamma + \delta \sinh^{-1} z)^2\right)}{\xi\sqrt{2\pi}\sqrt{z^2+1}}$$ where $$z = \frac{x-\lambda}{\xi}$$ Note $\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right)$
Cumulative distribution function	$F(x) = N(\gamma + \delta \sinh^{-1} z)$
Mean	$\lambda - \xi\sqrt{w}\sinh\Omega$ where $w = \exp(\delta^{-2})$ and $\Omega = \gamma/\delta$
Variance	$\frac{\xi^2}{2}(w-1)(w\cosh(2\Omega)+1)$
Skewness	$-\frac{\xi^3\sqrt{w}(w-1)^2(w(w+2)\sinh(3\Omega)+3\sinh\Omega)}{4\sigma^3}$ where $\sigma = \sqrt{\frac{\xi^2}{2}(w-1)(w\cosh(2\Omega)+1)}$
(Excess) kurtosis	$\frac{\xi^4(w-1)^2(A+B+C)}{8\sigma^4} - 3$ where $A = w^2(w^4+2w^3+3w^2-3)\cosh(4\Omega)$ $B = 4w^2(w+2)\cosh(2\Omega)$ $C = 3(2w+1)$

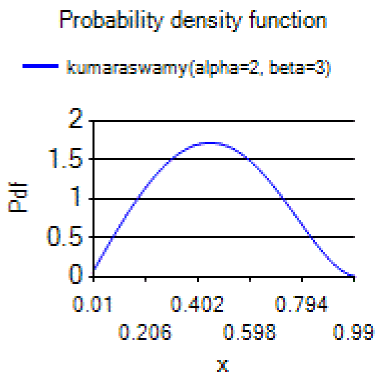
Nematrian web functions

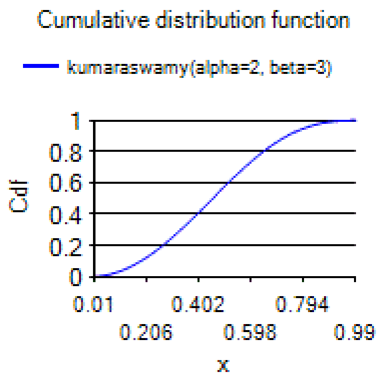
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “johnsonsu”. For details of other supported probability distributions see here.

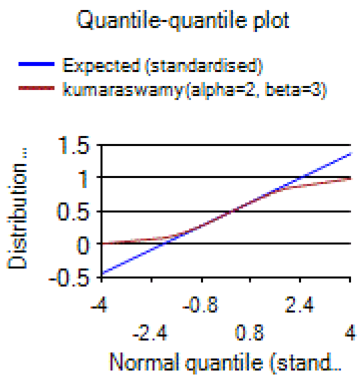
The Kumaraswamy distribution

[KumaraswamyDistribution]

The Kumaraswamy distribution describes a distribution in which outcomes are limited to a specific range, the probability density function within this range being characterised by two shape parameters. It is similar to the beta distribution but possibly easier to use because it has simpler analytical expressions for its probability density function and cumulative distribution function.







Distribution name	**Kumaraswamy distribution**
Common notation	$X \sim Kumaraswamy(\alpha, \beta)$
Parameters	α = shape parameter ($\alpha > 0$) β = shape parameter ($\beta > 0$)
Domain	$0 \leq x \leq 1$
Probability density function	$f(x) = \alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}$
Cumulative distribution function	$F(x) = 1-(1-x^\alpha)^\beta$
Mean	$\beta B\left(1+\frac{1}{\alpha}, \beta\right)$
Variance	$\left(\beta B\left(1+\frac{2}{\alpha}, \beta\right) - \beta^2 B\left(1+\frac{1}{\alpha}, \beta\right)^2\right)$
Other comments	A standard Kumaraswamy distribution has $a=0$ and $b=1$. Its non-central moments are given by: $$E(X^r) = \frac{\beta\Gamma(1+r/\alpha)\Gamma(\beta)}{\Gamma(1+\beta+r/\alpha)} = \beta B\left(1+\frac{r}{\alpha}, \beta\right)$$ and its median is $\left(1-2^{-1/\beta}\right)^{1/\alpha}$ and its mode is $\left(\frac{\alpha-1}{\alpha\beta-1}\right)^{1/\alpha}$ (for $\alpha \geq 1, \beta \geq 1, (\alpha,\beta) \neq (1,1)$).

<u>Nematrian web functions</u>

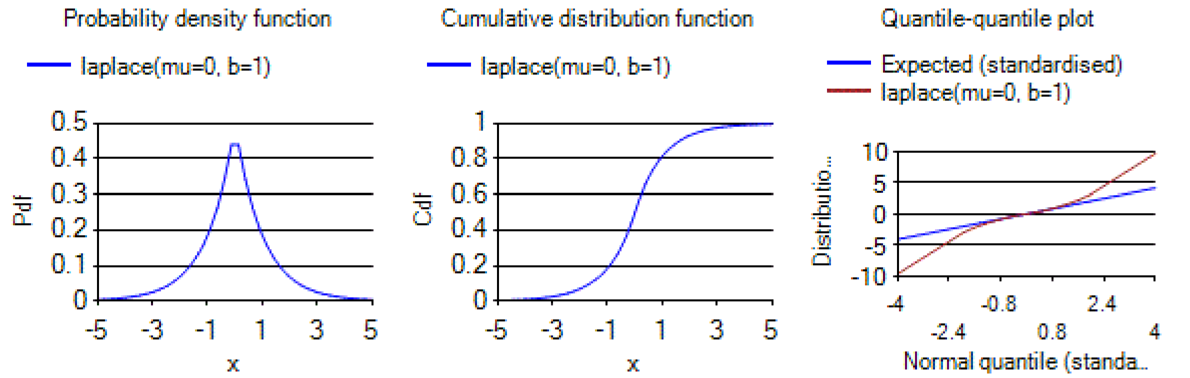
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “kumaraswamy”. Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a

DistributionName of "kumaraswamy4" ", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Laplace distribution

[LaplaceDistribution]

The Laplace distribution is akin to two exponential distributions spliced together.



Distribution name	**Laplace distribution**
Common notation	$Y \sim Laplace(\mu, b)$
Parameters	μ = location parameter b = scale parameter ($b > 0$)
Domain	$-\infty < x < +\infty$
Probability density function	$f(x) = \frac{1}{2b} \exp\left(-\frac{\lvert x-\mu \rvert}{b}\right)$
Cumulative distribution function	$F(x) = \begin{cases} \frac{1}{2}\exp\left(\frac{x-\mu}{b}\right) & x \le \mu \\ 1 - \frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right) & x > \mu \end{cases}$
Mean	μ
Variance	$2b^2$
Skewness	0
(Excess) kurtosis	3
Characteristic function	$\frac{e^{i\mu t}}{1+b^2t^2}$
Other comments	The median and mode are μ. The inverse cdf (i.e. quantile) function is $F^{-1}(p) = \mu - b\,\mathrm{sgn}\left(p-\frac{1}{2}\right)\log\left(1-2\left\lvert p-\frac{1}{2}\right\rvert\right)$. If $X \sim U\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $Y = \mu - b\,\mathrm{sgn}(X)\log(1-2\lvert X \rvert)$ then $Y \sim Laplace(\mu, b)$. It is sometimes referred to as the *double exponential* distribution (but this term is also apparently also used sometimes of the Gumbel distribution)

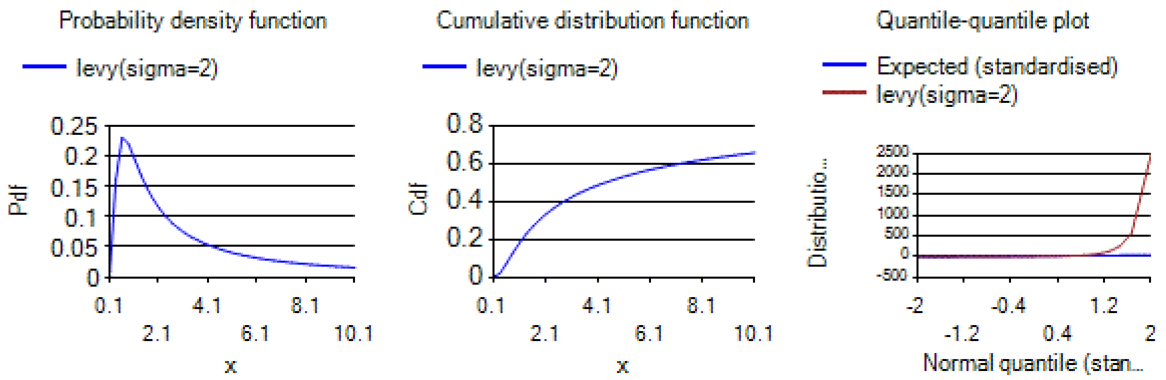
Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “laplace”. For details of other supported probability distributions see here.

The Lévy distribution

[LevyDistribution]

The time of hitting a single point a (different from the starting point of 0) of a Brownian motion without a mean drift follows the Lévy distribution with $\sigma = a^2$. With a mean drift it follows an inverse Gaussian meaning that the Lévy distribution is a special case of the inverse Gaussian distribution.



Distribution name	**Lévy distribution**
Common notation	$X \sim Levy(\sigma)$
Parameters	σ = scale parameter ($\sigma > 0$)
Domain	$0 < x < +\infty$
Probability density function	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{\exp\left(\frac{\sigma}{2x}\right)}{(x/\sigma)^{3/2}}$
Cumulative distribution function	$F(x) = 2 - 2N\left(\sqrt{\frac{\sigma}{x}}\right)$
Mean	∞
Variance	∞
Skewness	undefined
(Excess) kurtosis	undefined
Characteristic function	$e^{-\sqrt{-2i\sigma t}}$

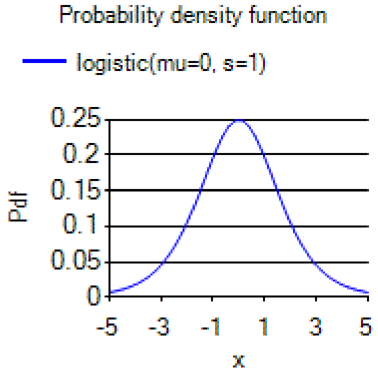
Nematrian web functions

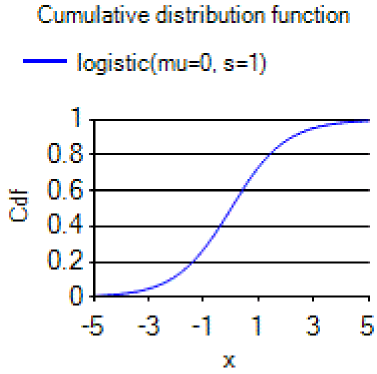
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “levy”. Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of “levy2”, see also including additional shift and scale parameters. For details of other supported probability distributions see here.

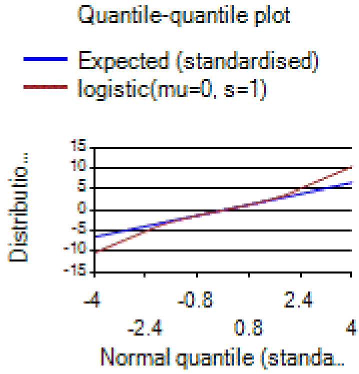
The logistic distribution

[LogisticDistribution]

The logistic distribution has a similar shape to the normal distribution but has heavier tails.







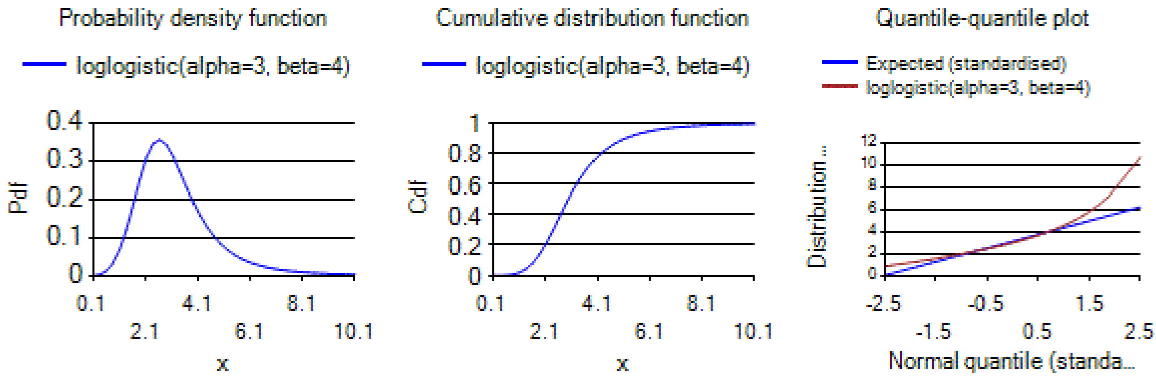
Distribution name	**Logistic distribution**
Common notation	$X \sim Logistic(\mu, s)$
Parameters	μ = location parameter s = scale parameter ($\sigma > 0$)
Domain	$-\infty < x < +\infty$
Probability density function	$$f(x) = \frac{\exp(-z)}{s(1+\exp(-z))^2}$$ where $$z = \frac{x-\mu}{s}$$
Cumulative distribution function	$$F(x) = \frac{1}{1+\exp(-z)}$$
Mean	μ
Variance	$\frac{\pi^2 s^2}{3}$
Skewness	0
(Excess) kurtosis	$\frac{6}{5}$
Characteristic function	$\frac{e^{\mu i t}\pi s t}{\sinh(\pi s t)}$
Other comments	The logistic distribution is sometimes called the *sech-square(d)* distribution because its pdf can also be expressed in terms of the hyperbolic secan function: $$f(x) = \frac{\exp(-z)}{\sigma(1+\exp(-z))^2} = \frac{1}{4\sigma}\text{sech}^2\left(\frac{z}{2}\right)$$ It should not then be confused with the hyperbolic secant distribution.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "logistic". For details of other supported probability distributions see here.

The log-logistic distribution

[LogLogisticDistribution]



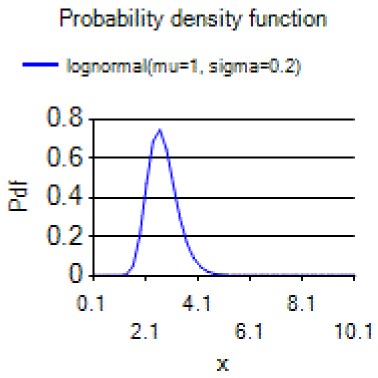
Distribution name	**Log-logistic distribution**
Common notation	$X \sim LL(\alpha, \beta)$
Parameters	α = scale parameter ($\alpha > 0$) β = shape parameter ($\beta > 0$)
Domain	$0 \leq x < +\infty$
Probability density function	$f(x) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^2}$
Cumulative distribution function	$F(x) = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}$
Mean	$\alpha b \csc(b) \quad if\ \beta > 1$ where $b = \pi/\beta$ and $\csc x = 1/\sin x$ is the cosecant function
Variance	$\alpha^2(2b\csc(2b) - b^2\csc^2 b) \quad if\ \beta > 2$
Other comments	Is also called the Fisk distribution. Its non-central moments are (if $k < \beta$): $E(X^r) = \alpha^k B(1-k/\beta, 1+k/\beta) = \frac{\alpha^k kb}{\sin(kb)}$

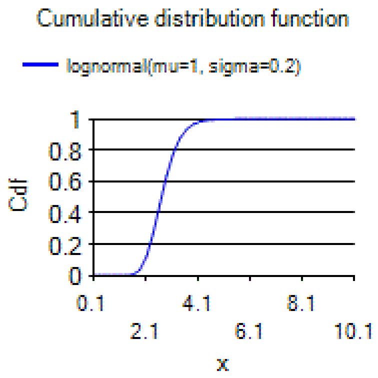
Nematrian web functions

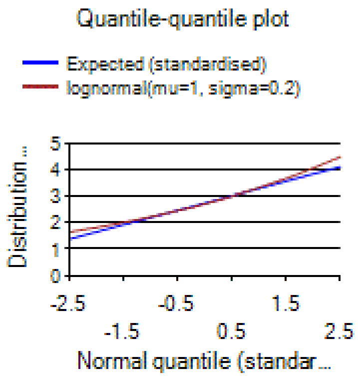
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "loglogistic". Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of "loglogistic3", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The lognormal distribution

[LognormalDistribution]







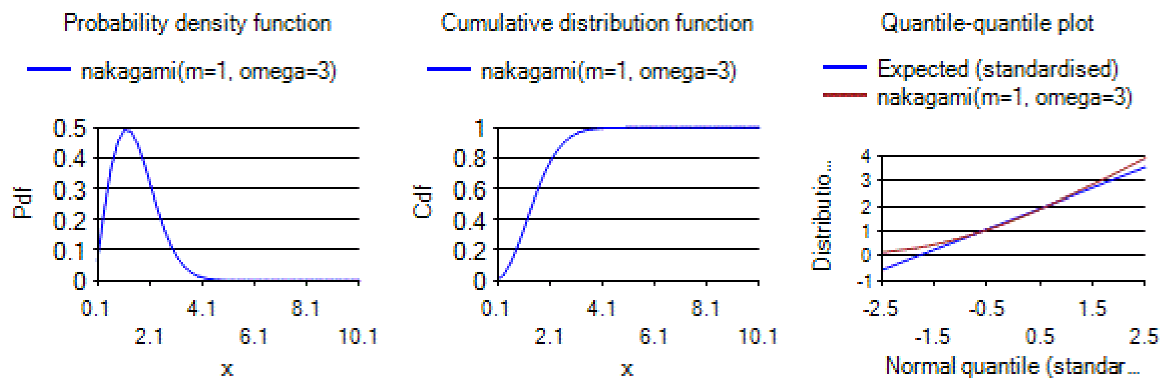
Distribution name	**Lognormal distribution**
Common notation	$X \sim logN(\mu, \sigma^2)$
Parameters	σ = scale parameter ($\sigma > 0$) μ = location parameter
Domain	$0 < x < +\infty$
Probability density function	$f(x) = \dfrac{\exp\left(-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right)}{x\sigma\sqrt{2\pi}}$
Cumulative distribution function	$F(x) = N\left(\dfrac{\log x - \mu}{\sigma}\right) = \dfrac{1}{2} + \dfrac{1}{2} erf\left(\dfrac{\log x - \mu}{\sqrt{2\sigma^2}}\right)$
Mean	$e^{\mu + \sigma^2/2}$
Variance	$\left(e^{(\sigma^2)} - 1\right)e^{2\mu + \sigma^2}$
Skewness	$\left(e^{(\sigma^2)} + 2\right)\sqrt{e^{(\sigma^2)} - 1}$
(Excess) kurtosis	$e^{(4\sigma^2)} + 2e^{(3\sigma^2)} + 3e^{(2\sigma^2)} - 6$
Characteristic function	No simple expression that is not divergent
Other comments	The median of a lognormal distribution is e^{μ} and its mode is $e^{\mu - \sigma^2}$. The truncated moments of $logN(\mu, \sigma^2)$ are: $\int_L^U x^k f(x)dx = e^{k\mu + k^2\sigma^2/2}\left(N\left(\dfrac{\log U - \mu}{\sigma} - k\sigma\right) - N\left(\dfrac{\log L - \mu}{\sigma} - k\sigma\right)\right)$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "lognormal". Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of "lognormal4" ", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Nakagami distribution

[LognormalDistribution]



Distribution name	**Nakagami distribution**
Common notation	$X \sim Nakagami(m, \omega)$
Parameters	m = parameter ($m \geq 1/2$) ω = parameter ($\omega > 0$)
Domain	$0 \leq x < +\infty$
Probability density function	$f(x) = \frac{2m^m}{\Gamma(m)\omega^m} x^{2m-1} \exp\left(-\frac{m}{\omega}x^2\right)$
Cumulative distribution function	$F(x) = \frac{\Gamma_{mx^2/\omega}(m)}{\Gamma(m)}$
Mean	$\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)}\left(\frac{\omega}{m}\right)^{1/2}$
Variance	$\omega\left(1-\frac{1}{m}\left(\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)}\right)^2\right)$
Other comments	Its median is $\sqrt{\omega}$ and its mode is $\frac{1}{\sqrt{2}}\left(\frac{(2m-1)\omega}{m}\right)^{1/2}$. If $X \sim \Gamma(k, \theta)$ then $Y = \sqrt{X} \sim Nakagami(k, k\theta)$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "nakagami". Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of "nakagami3", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The non-central chi-squared distribution

[NoncentralChiSquaredDistribution]

Distribution name	**Non-central chi-squared distribution**
Common notation	$X \sim \chi^2(\upsilon, \lambda)$
Parameters	ν = degrees of freedom (positive integer) λ = non-centrality parameter ($\lambda \geq 0$)
Domain	$0 \leq x < +\infty$

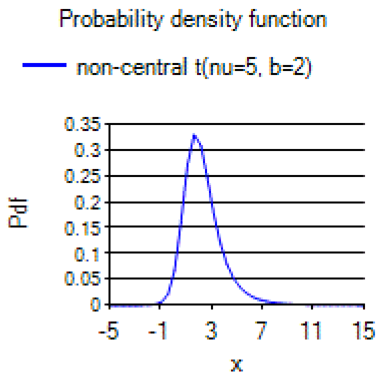
Probability density function	$$f(x) = \frac{x^{\nu/2-1}\exp\left(-\frac{x}{2}\right)}{2^{\nu/2}\Gamma(\nu/2)}\ {}_0F_1(;k/2;\lambda x/4)$$
Cumulative distribution function	$$F(x) = 1 - Q_{\nu/2}(\sqrt{\lambda}, \sqrt{x})$$ where $Q_M(a,b)$ is the Marcum-Q function.
Mean	$$\nu + \lambda$$
Variance	$$2(\nu + 2\lambda)$$
Skewness	$$\frac{2\sqrt{2}(\nu + 3\lambda)}{(\nu + 2\lambda)^{3/2}}\sqrt{\frac{8}{\nu}}$$
(Excess) kurtosis	$$\frac{12(\nu + 4\lambda)}{(\nu + 2\lambda)^2}$$
Characteristic function	$$(1 - 2it)^{-\nu/2}\exp\left(-\frac{i\lambda t}{1 - 2it}\right)$$
Other comments	The non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter λ is the distribution of the sum of the squares of ν independent normal distributions each with unit standard deviation but with non-zero means μ_i where $\lambda = \sum_{i=1}^{\nu}\mu_i^2$. The (standard) chi-squared distribution is a special case of it with $\lambda = 0$.

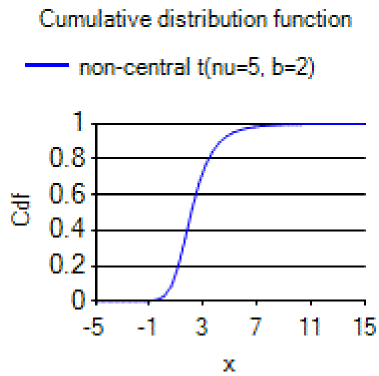
Nematrian web functions

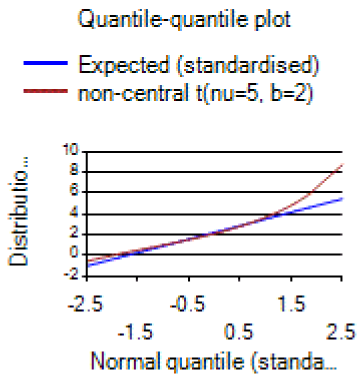
This distribution is not currently supported within the Nematrian web function library. For details of other supported probability distributions see here.

The non-central *t* distribution

[LognormalDistribution]







Distribution name	**(Standard) non-central *t* distribution (NCT)**
Common notation	$X \sim NCT(\nu, d)$
Parameters	ν = degrees of freedom ($\nu > 0$, usually ν is an integer although in some situations a non-integral ν can arise) b = non-centrality parameter
Domain	$-\infty < x < +\infty$

Probability density function	$$f(x)=\begin{cases}\frac{\upsilon}{x}\left(F_{\upsilon+2,b}\left(x\sqrt{1+2/\upsilon}\right)-F_{\upsilon,b}(x)\right) & if\ x\neq 0\\ \frac{\Gamma((\upsilon+1)/2)}{\sqrt{\pi\upsilon}\Gamma(\upsilon/2)}\exp\left(-\frac{b^2}{2}\right) & if\ x=0\end{cases}$$ where $F_{\upsilon,d}(x)$ is the NCT cumulative distribution function
Cumulative distribution function	$$F_{\upsilon,b}(x)=\begin{cases}Q_{\upsilon,b}(x) & if\ x\geq 0\\ 1-Q_{\upsilon,-b}(-x) & if\ x<0\end{cases}$$ where $$Q_{\upsilon,d}(x)=N(-b)+\frac{1}{2}\sum_{j=0}^{\infty}\left(p_jI_y\left(j+\frac{1}{2},\frac{\upsilon}{2}\right)+q_jI_y\left(j+1,\frac{\upsilon}{2}\right)\right)$$ $$y=\frac{x^2}{x^2+\upsilon}$$ $$p_j=\frac{1}{j!}\left(\frac{b^2}{2}\right)^j\exp\left(-\frac{b^2}{2}\right)$$ $$q_j=\frac{\mu}{\sqrt{2}\Gamma\left(j+\frac{3}{2}\right)}\left(\frac{b^2}{2}\right)^j\exp\left(-\frac{b^2}{2}\right)$$
Mean	$$b\sqrt{\frac{\nu}{2}}\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\quad if\ \nu>1$$
Variance	$$\frac{\nu(1+b^2)}{\nu-2}-\frac{b^2\nu}{2}\left(\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right)^2\quad if\ \nu>2$$
Other comments	X has a standard non-central t distribution with υ degrees of freedom and non-centrality parameter d if $X=(Z+b)/\sqrt{Y/\nu}$, $Y\sim\chi_\nu^2$, $Z\sim N(0,1)$ and Y and Z are independent. It has the following non-central moments: $$E(X^k)=\left(\frac{\nu}{2}\right)^{\frac{k}{2}}\frac{\Gamma\left(\frac{\nu-k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}e^{-b^2/2}\frac{d^ke^{b^2/2}}{db}\quad if\ \nu>k$$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "non-central t". Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of "non-central t4", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The normal distribution

[NormalDistribution]

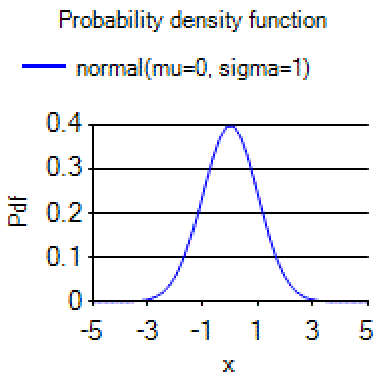
The normal distribution is a continuous probability distribution that has a bell-shaped probability density function:

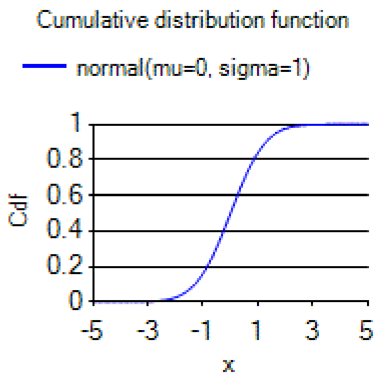
$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

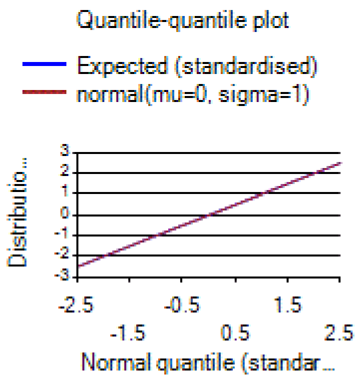
It is usually considered to be the most prominent probability distribution in statistics partly because it arises in a very large number of contexts as a result of the central limit theorem and partly because it is relatively tractable analytically.

The normal distribution is also called the *Gaussian* distribution. The *unit normal* (or *standard normal*) distribution is $N(0,1)$.

Characteristics of the normal distribution are set out below:







Distribution name	**Normal distribution**
Common notation	$X \sim N(\mu, \sigma^2)$
Parameters	σ = scale parameter ($\sigma > 0$) μ = location parameter
Domain	$-\infty < x < +\infty$
Probability density function	$f(x) \equiv \phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$
Cumulative distribution function	$F(x) \equiv N(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) dt$
Mean	μ
Variance	σ^2
Skewness	0
(Excess) kurtosis	0
Characteristic function	$e^{it\mu - \frac{1}{2}\sigma^2 t^2}$
Other comments	The inverse unit normal distribution function (i.e. its quantile function) is commonly written $N^{-1}(x)$ (also in some texts $\Phi(x)$ and the unit normal density function is commonly written $\phi(x)$. $N^{-1}(x)$ is also called the *probit* function. The error function distribution is $N\left(0, \frac{1}{2h}\right)$, where h is now an inverse scale parameter $h > 0$. The median and mode of a normal distribution are μ. The truncated first moments of $N(\mu, \sigma^2)$ are:

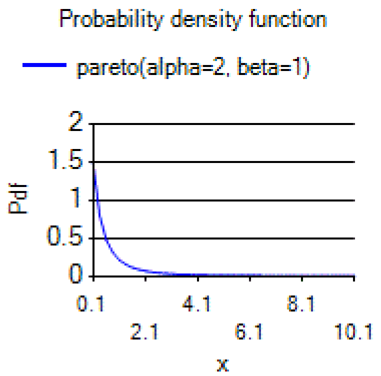
	$$\int_L^U xf(x)dx = \mu\left(N\left(\frac{U-\mu}{\sigma}\right) - N\left(\frac{L-\mu}{\sigma}\right)\right) - \sigma\left(\phi\left(\frac{U-\mu}{\sigma}\right) - \phi\left(\frac{L-\mu}{\sigma}\right)\right)$$ where $\phi(x)$ and $N(x)$ are the pdf and cdf of the unit normal distribution respectively. The mean excess function of a standard normal distribution is thus $$e(u) = \frac{\phi(u) - uN(-u)}{N(-u)} = \frac{\frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}u^2\right) - uN(-u)}{N(-u)}$$ The central moments of the normal distribution are: $$E\left((X-\mu)^k\right) = \begin{cases} 0 & if\ k\ is\ odd \\ \sigma^p \times 1 \times 3 \times \ldots \times (k-1) & if\ k\ is\ even \end{cases}$$

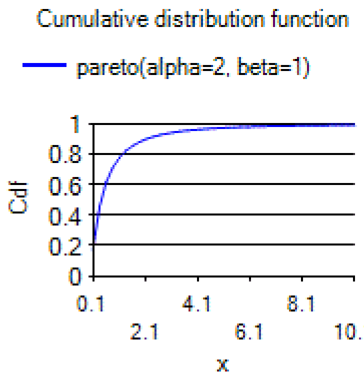
<u>Nematrian web functions</u>

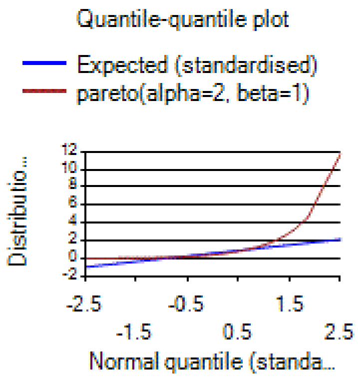
Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "normal". For details of other supported probability distributions see here.

The Pareto distribution

[ParetoDistribution]







Distribution name	**Pareto distribution**
Common notation	$X \sim Pareto(\alpha, \beta)$
Parameters	α = shape parameter ($\alpha > 0$) β = scale parameter ($\beta > 0$)
Domain	$0 \leq x < +\infty$
Probability density function	$f(x) = \frac{\alpha\beta^\alpha}{(\beta + x)^{\alpha+1}}$
Cumulative distribution function	$F(x) = 1 - \left(\frac{\beta}{\beta + x}\right)^\alpha$
Mean	$\frac{\beta}{\alpha - 1} \quad for\ \alpha > 1$

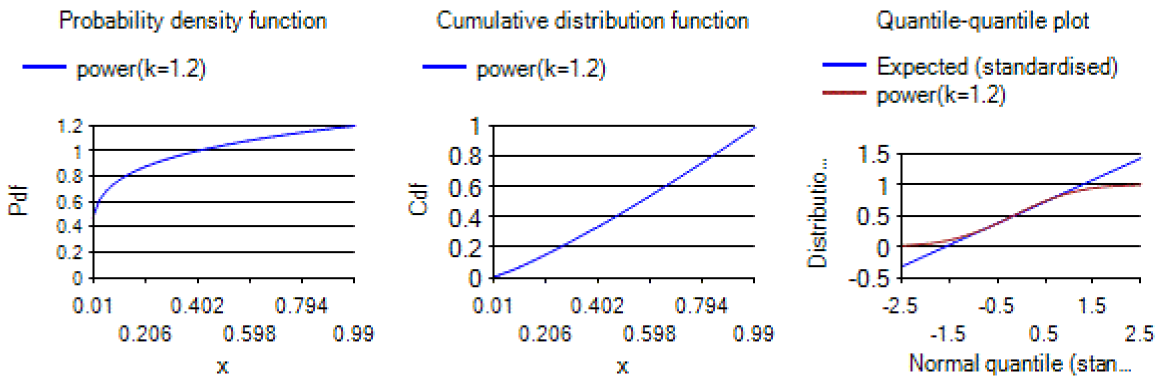
Variance	$$\frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)} \quad for\ \alpha > 2$$
Skewness	$$\frac{2(\alpha+1)}{\alpha-3}\sqrt{\frac{\alpha-2}{\alpha}} \quad for\ \alpha > 3$$
(Excess) kurtosis	$$\frac{6(\alpha^3+\alpha^2-6\alpha-2)}{\alpha(\alpha-3)(\alpha-4)} \quad for\ \alpha > 4$$
Other comments	Also known as the Pareto distribution of the second kind in which case the Pareto distribution of the first kind has $\beta \le x < +\infty$, $f(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}}$ and $F(x) = 1 - \left(\frac{\beta}{x}\right)^\alpha$.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "pareto". Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of "pareto3", see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The power function distribution

[PowerFunctionDistribution]



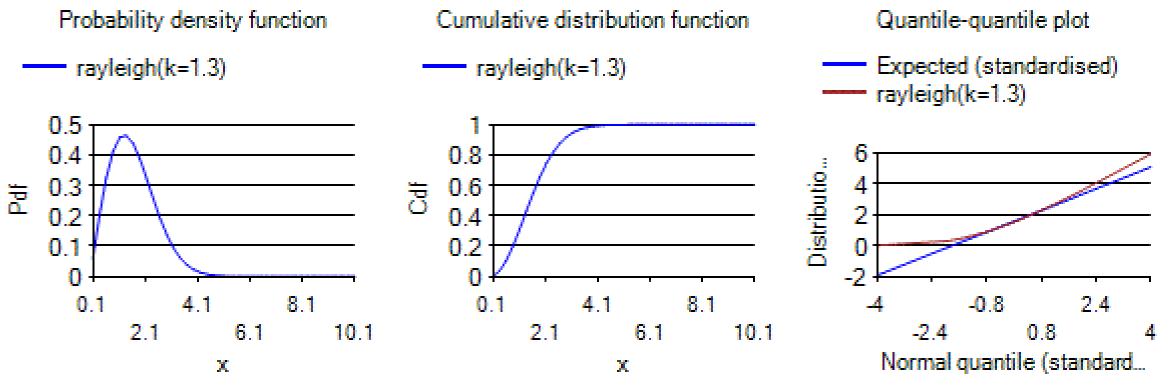
Distribution name	**Power function distribution**
Common notation	
Parameters	k = shape parameter ($k > 0$)
Domain	$0 \le x \le 1$
Probability density function	$f(x) = kx^{k-1}$
Cumulative distribution function	$F(x) = x^k$
Mean	$$\frac{k}{k+1}$$
Variance	$$\frac{k}{k^3+4k^2+5k+2}$$
Other comments	Its non-central moments around a (which can be used to derive its non-central moments around 0) are: $$E(X^r) = \frac{k}{r+k}$$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “power”. Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a *DistributionName* of “power3” ”, see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The Rayleigh distribution

[RayleighDistribution]



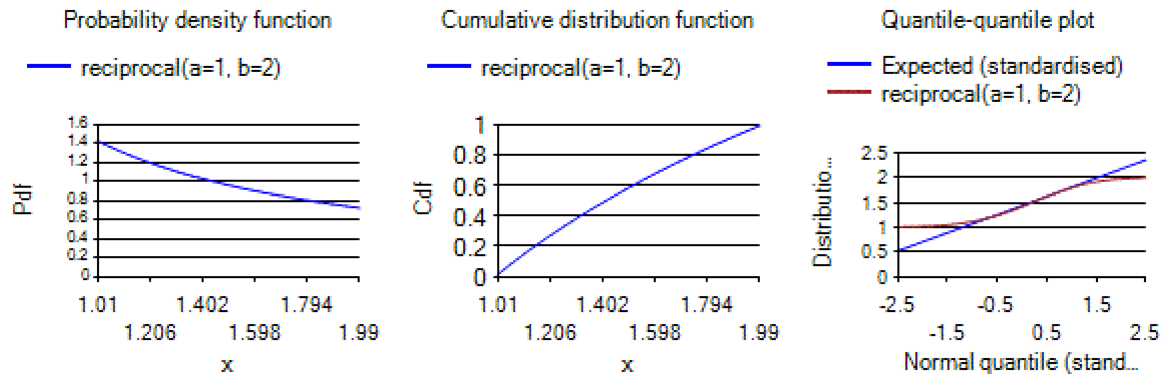
Distribution name	**Rayleigh distribution**
Common notation	$X \sim Rayleigh(k)$
Parameters	k = scale parameter ($k > 0$)
Domain	$0 \leq x < +\infty$
Probability density function	$f(x) = \dfrac{x \exp\left(-\dfrac{x^2}{2k^2}\right)}{k^2}$
Cumulative distribution function	$F(x) = 1 - \exp\left(-\dfrac{x^2}{2k^2}\right)$
Mean	$k\sqrt{\dfrac{\pi}{2}}$
Variance	$\dfrac{4-\pi}{2}k^2$
Skewness	$\dfrac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^2}$
(Excess) kurtosis	$-\dfrac{6\pi^2 - 24\pi + 16}{(4-\pi)^2}$
Characteristic function	$1 - kt\exp\left(-\dfrac{k^2t^2}{2}\right)\sqrt{\dfrac{\pi}{2}}\left(-i\,\mathrm{erf}\left(\dfrac{ikt}{\sqrt{2}}\right) - i\right)$ Here $\mathrm{erf}(z)$ is the error function, which is defined as: $\mathrm{erf}(z) = \dfrac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt \quad \Rightarrow \quad \mathrm{erf}(x) = 2.N(x\sqrt{2}) - 1$
Other comments	The Rayleigh distribution often arises when the overall magnitude of a vector is related to its directional components

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “rayleigh”. Functions relating to a generalised version of this distribution including an additional location (i.e. shift) parameter may be accessed by using a *DistributionName* of “rayleigh2”, see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The reciprocal distribution

[ReciprocalDistribution]



Distribution name	**Reciprocal distribution**
Common notation	
Parameters	a, b = boundary parameters ($0 < a < b$)
Domain	$a \le x \le b$
Probability density function	$f(x) = \dfrac{1}{x(\log b - \log a)}$
Cumulative distribution function	$F(x) = \dfrac{\log x - \log a}{\log b - \log a}$
Mean	$\dfrac{b-a}{\log b - \log a}$
Variance	$\dfrac{1}{2}\left(\dfrac{b^2 - a^2}{\log b - \log a}\right) - \left(\dfrac{b-a}{\log b - \log a}\right)^2$
Other comments	Its non-central moments are: $E(X^r) = \dfrac{b^r - a^r}{r(\log b - \log a)}$

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “reciprocal”. For details of other supported probability distributions see here.

The Rice distribution

[RiceDistribution]

The Rice distribution is characterises the magnitude of a circular bivariate normal random vector with potentially non-zero mean.

Distribution name	**Rice distribution**
Common notation	$X \sim Rice(\nu,\sigma)$
Parameters	ν = parameter σ = parameter
Domain	$0 \le x < +\infty$
Probability density function	$$f(x)=\frac{x}{\sigma^2}\exp\left(-\frac{x^2+\nu^2}{2\sigma^2}\right)I_0\left(\frac{x\nu}{\sigma^2}\right)$$ where $I_0(q)$ is the modified Bessel function of the first kind of order zero, i.e. $I_0(q)=\sum_{k=0}^{\infty}\frac{\left(\frac{q}{2}\right)^{2k}}{(k!)^2}$
Cumulative distribution function	$$F(x)=1-Q_1\left(\frac{\nu}{\sigma},\frac{x}{\sigma}\right)$$ where $Q_1(p,q)$ is the Marcum Q-function, i.e. $Q_1(p,q)=\int_q^{\infty} x\exp\left(-\frac{x^2+p^2}{2}\right)I_0(px)dx$.
Mean	$$\sigma\sqrt{\pi/2}\,L_{1/2}\left(-\frac{\nu^2}{2\sigma^2}\right)$$
Variance	$$2\sigma^2+\nu^2-\frac{\pi\sigma^2}{2}\left(L_{1/2}\left(-\frac{\nu^2}{2\sigma^2}\right)\right)^2$$
Other comments	The Rayleigh distribution is a special case of the Rice distribution where $\nu=0$. Its non-central moments are: $$E(X^r)=\sigma^r2^{r/2}\Gamma\left(1+\frac{r}{2}\right)L_{r/2}\left(-\frac{\nu^2}{2\sigma^2}\right) \qquad r=1,2,\ldots$$ where $L_q(x)= {}_1F_1(-q;1;x)$ is a Laguerre polynomial.

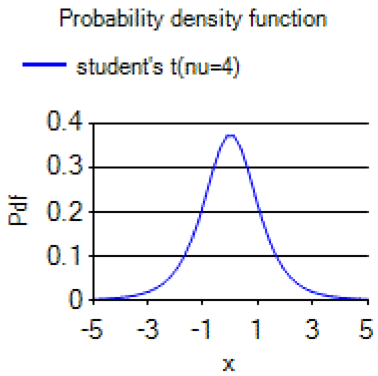
Nematrian web functions

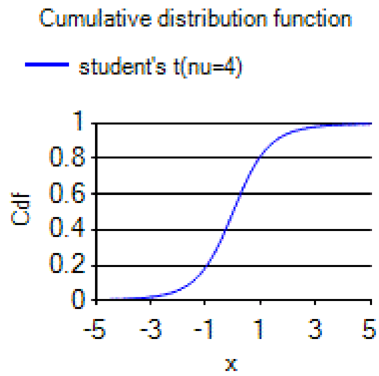
This distribution is not currently supported within the Nematrian web function library. For details of supported probability distributions see here.

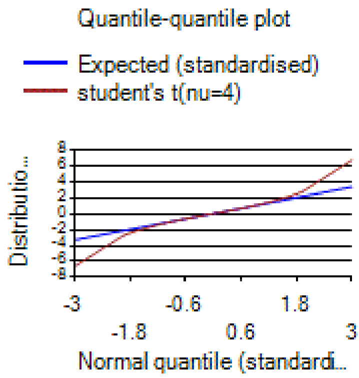
The Student's *t* distribution

[StudentsTDistribution]

The *Student's t* distribution (more simply the *t* distribution) arises when estimating the mean of a normally distributed population when sample sizes are small and the population standard deviation is unknown.







Distribution name	**(Standard) Student's *t* distribution**
Common notation	$X \sim t_\nu$
Parameters	ν = degrees of freedom ($\nu > 0$, usually ν is an integer although in some situations a non-integral ν can arise)
Domain	$-\infty < x < +\infty$
Probability density function	$$f(x) = \frac{1}{\sqrt{\pi\nu}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \frac{1}{\sqrt{\nu} B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$
Cumulative distribution function	$$F(x) = \begin{cases} \frac{1}{2} I_z\left(\frac{\nu}{2}, \frac{1}{2}\right) & x < 0 \\ 1 - \frac{1}{2} I_z\left(\frac{\nu}{2}, \frac{1}{2}\right) & x \geq 0 \end{cases}$$ where $z = \nu/(\nu + x^2)$
Mean	0
Variance	$$\frac{\nu}{\nu - 2} \quad for\ \ \nu > 2$$
Skewness	$0 \quad for\ \nu > 3$
(Excess) kurtosis	$$\frac{3(\nu - 2)}{\nu - 4} - 3 = \frac{6}{\nu - 4} \quad for\ \ \nu > 4$$
Characteristic function	$$\frac{K_{\nu/2}\left(\sqrt{\nu}\lvert t\rvert\right)\left(\sqrt{\nu}\lvert t\rvert\right)^{\nu/2}}{\Gamma(\nu/2) 2^{\nu/2-1}}$$ where $K_{\nu/2}(x)$ is a Bessel function
Other comments	It is a special case of the generalised hyperbolic distribution. Its non-central moments if r is even and $0 < r < \nu$ are: $$E(X^r) = \frac{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{\nu-r}{2}\right)\nu^{r/2}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}$$ If r is even and $0 < \nu \leq r$ then $E(X^r) = \infty$, if r is odd and $0 < r < \nu$ then $E(X^r) = 0$ and if r is odd and $0 < \nu \leq r$ then $E(X^r)$ is undefined.

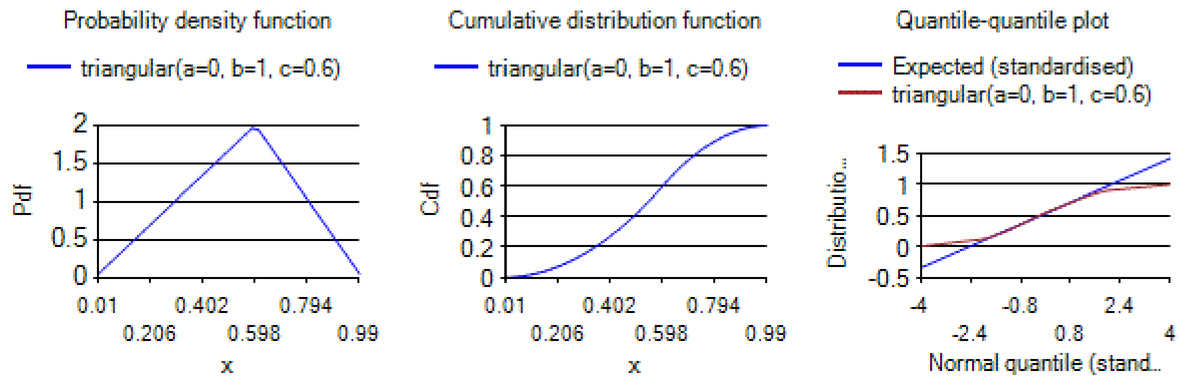
Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "student's t". Functions relating to a generalised version of this distribution including additional location (i.e. shift) and scale parameters may be accessed by using a

DistributionName of “student's t3”, see also including additional shift and scale parameters. For details of other supported probability distributions see here.

The triangular distribution

[TriangularDistribution]



Distribution name	**Triangular distribution**
Common notation	$X \sim Triangular(a,b,c)$
Parameters	a, b = boundary parameters ($a < b$) c = mode parameter ($a \leq c \leq b$)
Domain	$a \leq x \leq b$
Probability density function	$f(x) = \begin{cases} \frac{2(x-a)}{(c-a)(b-a)} & a \leq x \leq c \\ \frac{2(b-x)}{(b-c)(b-a)} & c < x \leq b \end{cases}$
Cumulative distribution function	$F(x) = \begin{cases} \frac{(x-a)^2}{(c-a)(b-a)} & a \leq x \leq c \\ 1 - \frac{(b-x)^2}{(b-c)(b-a)} & c < x \leq b \end{cases}$
Mean	$\frac{a+b+c}{3}$
Variance	$\frac{a^2+b^2+c^2-ab-ac-bc}{18}$
Skewness	$\frac{\sqrt{2}(a+b-2c)(2a-b-c)(a-2b+c)}{5(a^2+b^2+c^2-ab-ac-bc)^{3/2}}$
(Excess) kurtosis	$-\frac{3}{5}$
Characteristic function	$-2\frac{(b-c)e^{iat}-(b-a)e^{ict}+(c-a)e^{ibt}}{(b-a)(c-a)(b-c)t^2}$
Other comments	Its mode is c. Its median is: $\begin{cases} a + \sqrt{\frac{(c-a)(b-a)}{2}} & c \geq \frac{a+b}{2} \\ b - \sqrt{\frac{(c-a)(b-a)}{2}} & c \leq \frac{a+b}{2} \end{cases}$

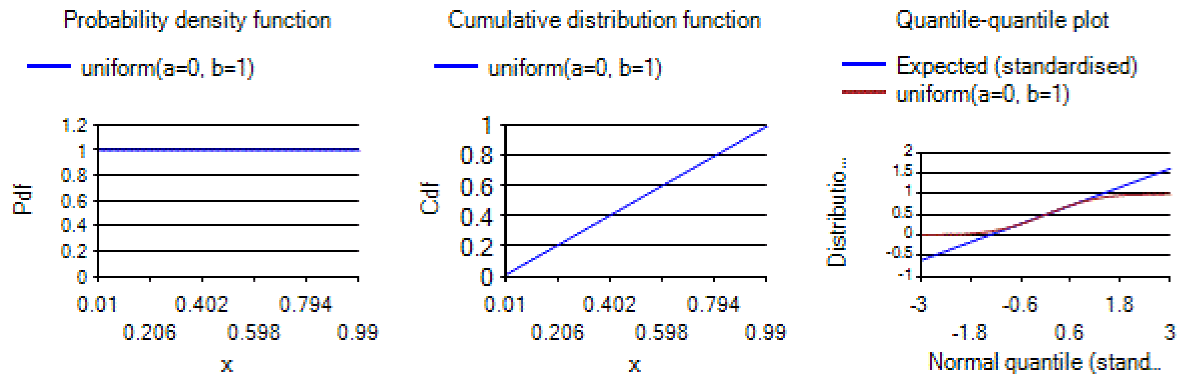
Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “triangular”. For details of other supported probability distributions see here.

The uniform distribution

[UniformDistribution]

The uniform distribution describes a (continuous) probability distribution in which any outcome within a given range is equally probable.



Distribution name	**Uniform distribution**
Common notation	$X \sim U(a,b)$
Parameters	a,b = boundary parameters ($a < b$)
Domain	$a \leq x \leq b$
Probability density function	$f(x) = \frac{1}{b-a}$
Cumulative distribution function	$F(x) = \frac{x-a}{b-a}$
Mean	$(a+b)/2$
Variance	$(b-a)^2/12$
Skewness	0
(Excess) kurtosis	$-6/5$
Characteristic function	$\frac{e^{ibt} - e^{iat}}{it(b-a)}$
Other comments	Its non-central moments $(r = 1,2,3,\ldots)$ are $E(X^r) = \frac{1}{(b-1)}\frac{1}{r+1}(b^{r+1} - a^{r+1})$. Its median is $(a+b)/2$.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of “uniform”. For details of other supported probability distributions see here.

The Weibull distribution

[WeibullDistribution]

The Weibull distribution is a special case of the generalised extreme value (GEV) distribution. It characterises the distribution of ‘block maxima’ under certain (relatively restrictive) conditions.

Continuous multivariate distributions

The inverse Wishart distribution

[InverseWishartDistribution]

The inverse Wishart distribution (otherwise called the inverted Wishart distribution) $W^{-1}(\mathbf{V},m)$ is a probability distribution that is used in the Bayesian analysis of real-valued positive definite matrices (e.g. matrices of the type that arise in risk management contexts). It is a conjugate prior for the covariance matrix of a multivariate normal distribution.

It has the following characteristics, where $\mathbf{B}$ is a $n \times n$ matrix, $\mathbf{V}$ is a positive definite matrix and $\Gamma_n(.)$ is the multivariate gamma function.

Parameters (and constraints on parameters):	$m > n - 1$ (m = degrees of freedom, real) $\mathbf{V} > 0$ ($\mathbf{V}$ = inverse scale matrix, positive definite)
Support (i.e. values that it can take)	$\mathbf{B} > 0$, i.e. is positive definite, $\mathbf{B}$ an $n \times n$ matrix
Probability density function	$\dfrac{\lvert\mathbf{V}\rvert^{m/2}\lvert\mathbf{B}\rvert^{-(m+n+1)/2}e^{-\text{trace}(\mathbf{VB}^{-1})/2}}{2^{mn/2}\Gamma_n(m/2)}$
Mean	$\dfrac{\mathbf{V}}{m-n-1}$

If the elements of $\mathbf{B}$ are $B_{i,j}$ and the elements of $\mathbf{V}$ are $V_{i,j}$ then

$$\text{var}(B_{i,j}) = \frac{(m-n+1)V_{i,j}^2 + (m-n-1)V_{i,i}V_{i,j}}{(m-n)(m-n-1)^2(m-n-3)}$$

The main use of the inverse Wishart distribution appears to arise in Bayesian statistics. Suppose we want to make an inference about a covariance matrix, $\mathbf{V}$, whose prior $p(\mathbf{V})$ has a $W^{-1}(\mathbf{V},m)$ distribution. If the observation set $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_K)$ where the $\mathbf{X}_k$ are independent n-variate Normal (i.e. Gaussian) random variables drawn from a $N(0,\mathbf{Q})$ distribution then the conditional distribution $p(\mathbf{Q}|\mathbf{X})$, i.e. the probability of $\mathbf{Q}$ given $\mathbf{X}$, has a $W^{-1}(\mathbf{A}+\mathbf{V}, m+K)$ distribution, where $\mathbf{A} = \mathbf{XX}^T$ is the sample covariance matrix.

The univariate special case of the inverse Wishart distribution is the inverse gamma distribution. With $n = 1$, $\alpha = m/2$, $\beta(= \mathbf{V}/2) = V_{1,1}/2$, $x(= \mathbf{B}/2) = B_{1,1}/2$ we have:

$$p(x|\alpha,\beta) = \frac{\beta^\alpha x^{-\alpha-1}\exp(-\beta/x)}{\Gamma(\alpha)}$$

where $\Gamma(.) \equiv \Gamma_1(.)$ is the ordinary (i.e. univariate) Gamma function, see MnGamma.

For other probability distributions see here.

Copulas (a copula is a special type of continuous multivariate distribution)

The Clayton copula

[ClaytonCopula]

The *Clayton* copula is a copula that allows any specific non-zero level of (lower) tail dependency between individual variables. It is an Archimedean copula, and exchangeable.

Copula name	**Clayton copula**
Common notation	$U \sim C_\theta^{Cl}$
Parameters	$\theta \geq 0$ (can be extended to $\theta \geq -1$)
Domain	$0 \leq u_i \leq 1 \quad i = 1, \dots, n$
Copula	$C_\theta^{Cl}(u_1, \dots, u_n) = \left(\sum_i (u_i^{-\theta} - 1) + 1 \right)^{-1/\theta}$ Or if $\theta = 0$ we use the limit $\lim_{\theta \to 0} C_\theta^{GC}$
Kendall's rank correlation coefficient (for bivariate case)	$\dfrac{\theta}{2 + \theta}$
Coefficient of upper tail dependence, λ_u	0
Coefficient of lower tail dependence, λ_l	$2^{-1/\theta}$
Archimedean generator function, $\phi(t)$	$\theta^{-1}(t^{-\theta} - 1)$ Or if $\theta = 0$ we use the limit $\lim_{\theta \to 0} \phi_\theta(t) = -\log t$. If $\theta \neq 0$ then a simpler version, which does not alter the copula itself, is $\phi(t) = t^{-\theta} - 1$.
Other comments	If $\theta = 0$ we obtain the independence copula. The Clayton copula (like the Frank copula) is a comprehensive copula in that it interpolates between a lower limit of the countermonotonicity copula ($\theta \to -1$) and an upper limit of the comonotonicity copula ($\theta \to +\infty$)..

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "Clayton Copula". For details of other probability distributions see here.

The comonotonicity copula

[ComonotonicityCopula]

The *Comonotonicity* copula is a special copula characterising perfect positive dependence, in the sense that the U_i are almost surely strictly increasing functions of each other.

Copula name	**comonotonicity copula**
Common notation	$U \sim M$
Parameters	θ

Domain	$0 \leq u_i \leq 1 \quad i = 1, \ldots, n$
Copula	$M(u_1, \ldots, u_n) = min(u_1, \ldots u_n)$
Kendall's rank correlation coefficient (for bivariate case)	1
Coefficient of upper tail dependence	1
Coefficient of lower tail dependence	1
Other comments	Is the extreme case where dependency is as strongly "positive" as possible (i.e. achieves the Fréchet upper bound)

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "Comonotonicity Copula". For details of other supported probability distributions see here.

The countermonotonicity copula

[CountermonotonicityCopula]

The *Countermonotonicity* copula is a special copula characterising perfect negative dependence, in the sense that U_1 is almost surely a strictly decreasing function of U_2 and vice-versa.

Copula name	**countermonotonicity copula**
Common notation	$U \sim W$
Parameters	θ
Domain	$0 \leq u_i \leq 1 \quad i = 1,2$
Copula	$W(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$
Kendall's rank correlation coefficient	-1
Coefficient of upper tail dependence	1
Coefficient of lower tail dependence	1
Other comments	Is the extreme case where dependency is as strongly "negative" as possible (i.e. achieves the Fréchet lower bound)

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "Countermonotonicity Copula". For details of other supported probability distributions see here.

The Frank copula

[FrankCopula]

The *Frank* copula is a copula that is sometimes used in the modelling of codependency. It is an Archimedean copula, and exchangeable.

Copula name	**Frank copula**
Common notation	$U \sim C_\theta^{Fr}$
Parameters	$-\infty < \theta < \infty$
Domain	$0 \leq u_i \leq 1 \qquad i = 1, \dots, n$
Copula	$C_\theta^{Fr}(u_1, \dots, u_n) = -\frac{1}{\theta}\log\left(1 + \frac{\prod_i(\exp(-\theta u_i) - 1)}{\exp(-\theta) - 1}\right)$ Or if $\theta = 0$ we use the limit $\lim_{\theta \to 0} C_\theta^{Fr}$ which the independence copula
Kendall's rank correlation coefficient (for bivariate case)	$1 - 4\theta^{-1}(1 - D_1(\theta))$ where $D_1(\theta)$ is the Debye function defined as: $D_1(\theta) = \theta^{-1}\int_0^\infty \frac{t}{\exp(t) - 1} dt$
Coefficient of upper tail dependence, λ_u	0
Coefficient of lower tail dependence, λ_l	0
Archimedean generator function, $\phi(t)$	$-\log\left(\frac{\exp(-\theta t) - 1}{\exp(-\theta) - 1}\right)$ Or if $\theta = 0$ we use the limit $\lim_{\theta \to 0} \phi_\theta(t)$ which is taken as $-\log t$
Other comments	If $\theta = 0$ we obtain the independence copula. The Frank copula (like the Clayton copula) is a *comprehensive* copula in that it interpolates between a lower limit of the countermonotonicity copula ($\theta \to -\infty$) and an upper limit of the comonotonicity copula ($\theta \to +\infty$).

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "Frank Copula". For details of other supported probability distributions see here.

The Generalised Clayton copula

[GeneralisedClaytonCopula]

The *Generalised Clayton* copula is a copula that allows any specific (non-zero) level of (lower) tail dependency between individual variables. It is an Archimedean copula and exchangeable.

Copula name	**Generalised Clayton copula**
Common notation	$U \sim C_{\theta,\delta}^{GC}$
Parameters	$\theta \geq 0, \delta \geq 1$
Domain	$0 \leq u_i \leq 1 \qquad i = 1, \dots, n$
Copula	$C_{\theta,\delta}^{GC}(u_1, \dots, u_n) = \left(\left(\sum_i (u_i^{-\theta} - 1)^\delta\right)^{1/\delta} + 1\right)^{-1/\theta}$ Or if $\theta = 0$ we use the limit $\lim_{\theta \to 0} C_\theta^{GC}$

Kendall's rank correlation coefficient (for bivariate case), ρ_τ	$\dfrac{(2+\theta)\delta-2}{(2+\theta)\delta}$
Coefficient of upper tail dependence, λ_u	$2-2^{-1/\delta}$
Coefficient of lower tail dependence, λ_l	$2^{-1/(\theta\delta)}$
Archimedean generator function, $\phi(t)$	$\theta^{-\delta}\left(t^{-\theta}-1\right)$ Or if $\theta=0$ we use the limit $\lim_{\theta\to 0}\phi_\theta(t)$
Other comments	When $\delta=1$ the generalised Clayton copula becomes the Clayton copula.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "Generalised Clayton Copula". For details of other supported probability distributions see here.

The Gumbel copula

[GumbelCopula]

The *Gumbel* copula is a copula that allows any specific level of (upper) tail dependency between individual variables. It is an Archimedean copula, and exchangeable.

Copula name	**Gumbel copula**
Common notation	$U \sim C_\theta^{Gu}$
Parameters	$1 \le \theta < \infty$
Domain	$0 \le u_i \le 1 \quad i=1,\dots,n$
Copula	$C_\theta^{Gu}(u_1,\dots,u_n)=\exp\left(-\left(\sum_i(-\log u_i)^\theta\right)^{1/\theta}\right)$
Kendall's rank correlation coefficient (for bivariate case)	$1-\dfrac{1}{\theta}$
Coefficient of upper tail dependence, λ_u	$2-2^{1/\theta}$
Coefficient of lower tail dependence, λ_l	0
Archimedean generator function, $\phi(t)$	$(-\log t)^\theta$
Other comments	If $\theta=1$ we obtain the independence copula and as $\theta\to\infty$ we approach the comonotonicity copula.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "Gumbel Copula". For details of other supported probability distributions see here.

The Gaussian copula

[GaussianCopula]

The *Gaussian* copula is the copula that underlies the multivariate normal distribution.

Copula name	**Gaussian copula**
Common notation	$U \sim C_C^{Ga}$
Parameters	C, a non-negative definite $n \times n$ matrix, i.e. a matrix that can correspond to a correlation matrix
Domain	$0 \le u_i \le 1 \quad i = 1, \ldots, n$
Copula	$C_C^{Ga}(u_1, \ldots, u_n) = \mathbf{\Phi}_C\left(N^{-1}(u_1), \ldots, N^{-1}(u_2)\right)$ where $N^{-1}(x)$ is the inverse normal function and $\mathbf{\Phi}_C$ is the cumulative distribution function of the multivariate normal distribution defined by a covariance matrix equal to C
Kendall's rank correlation coefficient (for bivariate case), ρ_τ	$\frac{2}{\pi} \arcsin \rho$ Where ρ is the correlation coefficient between the two variables
Coefficient of upper tail dependence, λ_u	0 (unless the correlation matrix exhibits perfect positive or negative dependence)
Coefficient of lower tail dependence, λ_l	0 (unless the correlation matrix exhibits perfect positive or negative dependence)
Other comments	The Spearman rank correlation coefficient is given by: $\rho_S = \frac{6}{\pi} \arcsin \frac{\rho}{2}$ where ρ is the (normal) correlation coefficient between the two variables. If $C = I_n$ (the $n \times n$ identity matrix) then we obtain the independence copula.

Nematrian web functions

Functions relating to the above distribution in the two dimensional case may be accessed via the Nematrian web function library by using a *DistributionName* of "Gaussian Copula (2d)". For details of other supported probability distributions see here.

The Independence copula

[IndependenceCopula]

The *Independence* copula is the copula that results from a dependency structure in which each individual variable is independent of each other. It is an Archimedean copula, and exchangeable.

Copula name	**Independence copula**
Common notation	$U \sim \Pi$
Parameters	None
Domain	$0 \le u_i \le 1 \quad i = 1, \ldots, n$
Copula	$\Pi(u_1, \ldots, u_n) = \prod_i u_i$

Kendall's rank correlation coefficient (for bivariate case)	0
Coefficient of upper tail dependence, λ_u	0
Coefficient of lower tail dependence, λ_l	0
Archimedean generator function, $\phi(t)$	$-\log t$
Other comments	The independence copula is a special case of several Archimedean copulas. It is also the special case of the Gaussian copula with a correlation matrix equal to the identity matrix.

Nematrian web functions

Functions relating to the above distribution may be accessed via the Nematrian web function library by using a *DistributionName* of "Independence Copula". For details of other supported probability distributions see here.

The *t* copula

[TCopula]

The *t* copula is the copula that underlies the multivariate Student's *t* distribution.

Copula name	***t* copula**
Common notation	$U \sim C_{v,C}^{t}$
Parameters	C, a non-negative definite $n \times n$ matrix, i.e. a matrix that can correspond to a correlation matrix v = degrees of freedom ($v > 0$, usually v is an integer although in some situations a non-integral v can arise) (note in principle each marginal distribution could in principle have a different number of degrees of freedom although such a refinement is not commonly seen)
Domain	$0 \leq u_i \leq 1 \qquad i = 1, \dots, n$
Copula	$C_{v,C}^{t}(u_1, \dots, u_n) = \boldsymbol{t}_{v,C}\left(t_v^{-1}(u_1), \dots, t_v^{-1}(u_2)\right)$ where $t_v^{-1}(x)$ is the inverse student's t function and $\boldsymbol{t}_{v,C}$ is the cumulative distribution function of the multivariate student's t distribution with arbitrary mean and matrix generator equal to C
Kendall's rank correlation coefficient (for bivariate case), ρ_τ	$\frac{2}{\pi} \arcsin \rho$ Where ρ is the correlation coefficient between the two variables
Coefficient of upper tail dependence, λ_u	$2t_{v+1}\left(-\sqrt{\frac{(v+1)(1-\rho)}{1+\rho}}\right)$
Coefficient of lower tail dependence, λ_l	$2t_{v+1}\left(-\sqrt{\frac{(v+1)(1-\rho)}{1+\rho}}\right)$
Other comments	If $C = I_n$ (the $n \times n$ identity matrix) then, in contrast to the Gaussian copula, we do *not* recover the independence copula.

Nematrian web functions

Functions relating to the above distribution in the two-dimensional case may be accessed via the Nematrian web function library by using a *DistributionName* of "student's t (2d)". For details of other supported probability distributions see here.