Maximum Likelihood Estimation of Normal Distribution

The probability density (likelihood) of a single variable drawn from a Normal distribution $N(\mu, \sigma^2)$ is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Thus the likelihood of n independent draws $x_1, \dots, x_n$ from this distribution is:

$$L(x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

To identify the values of μ and σ that maximise L it is easiest to identify the maximum of the log likelihood (as the two will be maximised for the same values of μ and σ since $\log x$ is a monotonically increasing function of x). The log likelihood is:

$$\log L(x_1, \dots, x_n) = -n \log \sigma - \frac{1}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

This is maximised when $\frac{\partial \log L}{\partial \mu} = 0$ and $\frac{\partial \log L}{\partial \sigma} = 0$, i.e. when:

$$\frac{\partial \log L}{\partial \mu} = -2\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML}) = 0 \quad \therefore \quad \hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\hat{\sigma}_{ML}} + \frac{1}{\hat{\sigma}_{ML}^3}\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML})^2 = 0 \quad \therefore \quad \hat{\sigma}_{ML} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML})^2}$$

The maximum likelihood estimator, $\hat{\mu}_{ML}$, of the mean μ is thus the average of the observations, $\bar{x}$. It is possible to show that this is also the minimum variance unbiased estimator of μ. The maximum likelihood estimator, $\hat{\sigma}_{ML}$, of σ is the *population* standard deviation, s_p of the x_i which can be determined using the MnPopulationStdev web function. Please note that whilst $\hat{\mu}_{ML}$ is an unbiased estimator of μ, $\hat{\sigma}_{ML}$ is a *biased* estimator of σ. The minimum variance *unbiased* estimator of σ^2 is the *sample* variance (i.e. square of the *sample* standard deviation). The sample standard deviation is:

$$\hat{\sigma} = s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML})^2}$$