# Marginal Value-at-Risk (Marginal VaR) when underlying distribution is multivariate normal



Suppose we have a set of $n$ risk factors which we can characterise by an $n$-dimensional vector $\mathbf{x} = (x_1, \dots, x_n)^T$. Suppose that the (active) exposures we have to these factors are characterised by another $n$-dimensional vector, $\mathbf{a} = (a_1, \dots, a_n)^T$. Then the aggregate exposure is $\mathbf{a}.\mathbf{x}$.

The *Tail Value-at-Risk*, $VaR_\alpha(\mathbf{a})$, of the portfolio of exposures $\mathbf{a}$ at confidence level $\alpha$, is defined as the value such that $Pr\big(\mathbf{a}.\mathbf{x} \leq -VaR_\alpha(\mathbf{a})\big) = 1 - \alpha$. The Marginal Value-at-Risk, $MVaR_{\alpha,i}(\mathbf{a})$, is the sensitivity of $VaR_\alpha(\mathbf{a})$ to a small change in $i$'th exposure, i.e.:

$$MVaR_{\alpha,i}(\mathbf{a}) = \frac{\partial VaR_\alpha(\mathbf{a})}{\partial a_i}$$

In the case where the risk factors are multivariate normally distributed with mean $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$ and covariance matrix $V$ whose elements are $V_{ij}$ we have the following.

As $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{V})$ we have $\mathbf{a}.\mathbf{x} \sim N(\mathbf{a}.\boldsymbol{\mu}, \mathbf{a}^T\mathbf{V}\mathbf{a})$ $\Rightarrow VaR_\alpha(\mathbf{a}) = -\big(\mathbf{a}.\boldsymbol{\mu} + \sigma N^{-1}(1-\alpha)\big)$ where $\sigma \equiv \sqrt{\mathbf{a}^T\mathbf{V}\mathbf{a}}$ is the standard deviation of the volatility of the (active) portfolio return, otherwise known if we are focusing on active exposures as the (ex-ante) *tracking error*.

$$\Rightarrow MVaR_{\alpha,i}(\mathbf{a}) \equiv \frac{\partial VaR_\alpha(\mathbf{a})}{\partial a_i} = -\frac{\partial}{\partial a_i}\left(\mathbf{a}.\boldsymbol{\mu} + N^{-1}(1-\alpha)\sqrt{\mathbf{a}^T\mathbf{V}\mathbf{a}}\right)$$

$$\Rightarrow MVaR_{\alpha,i}(\mathbf{a}) = -\left(\frac{\partial}{\partial a_i}\left(\sum_{j=1}^{n} a_j\mu_j\right) - N^{-1}(1-\alpha)\frac{1}{2\sqrt{\mathbf{a}^T\mathbf{V}\mathbf{a}}}\frac{\partial}{\partial a_i}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} a_j V_{jk} a_k\right)\right)$$

$$\Rightarrow MVaR_{\alpha,i}(\mathbf{a}) = -\left(\mu_i + N^{-1}(1-\alpha)\frac{1}{\sigma}\left(\sum_{j=1}^{n} a_j V_{ij}\right)\right)$$

The last part of this equation can be expressed in terms of the correlation between $x_i$ and $\mathbf{a}.\mathbf{x}$ as follows. Suppose we view the $x_i$ as corresponding to time series $x_{i,t}$ with $T$ elements (which without loss of generality can be assumed to be de-meaned, i.e. to have their means set to zero) and $\mathbf{a}.\mathbf{x}$ as corresponding to a time series $y_t = \sum_{i=1}^{T} a_i x_{i,t}$. Then the correlation between $x_i$ and $\mathbf{a}.\mathbf{x}$ would be (ignoring any small sample adjustment):

$$Correlation(x_i, \mathbf{a}.\mathbf{x}) = \frac{\frac{1}{T}\sum_{t=1}^{T} x_{i,t} y_t}{\sqrt{\frac{1}{T}\sum_{t=1}^{T} x_{i,t}^2 \frac{1}{T}\sum_{t=1}^{T} y_t^2}}$$

We would also have:

$$V_{ij} = \frac{1}{T}\sum_{t=1}^{T} x_{i,t} x_{j,t}$$

$$\frac{1}{T}\sum_{t=1}^{T} x_{i,t} y_t = \frac{1}{T}\sum_{t=1}^{T} x_{i,t} \sum_{j=1}^{n} a_j x_{j,t} = \frac{1}{T}\sum_{j=1}^{n} a_j \sum_{t=1}^{T} x_{i,t} x_{j,t} = \sum_{j=1}^{n} a_j V_{ij}$$

$$\sum_{t=1}^{T} x_{i,t}^2 = V_{ii}$$

$$\frac{1}{T}\sum_{t=1}^{T} y_t^2 = \frac{1}{T}\sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k \sum_{t=1}^{T} x_{i,t} x_{j,t} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_j V_{jk} a_k = \mathbf{a}^T \mathbf{V} \mathbf{a} = \sigma^2$$

$$\Rightarrow Correlation(x_i, \mathbf{a}.\mathbf{x}) = \frac{\sum_{j=1}^{n} a_j V_{ij}}{\sqrt{V_{ii}}\sigma}$$

$$\Rightarrow \sum_{j=1}^{n} a_j V_{ij} = Correlation(x_i, \mathbf{a}.\mathbf{x})\sqrt{V_{ii}}\sigma$$

$$\Rightarrow MVaR_{\alpha,i}(\mathbf{a}) = -\left(\mu_i + N^{-1}(1-\alpha)Correlation(x_i, \mathbf{a}.\mathbf{x})\sqrt{V_{ii}}\right)$$

As risks arising from individual positions interact there is no universally agreed way of subdividing the overall risk into contributions from individual positions. However, a commonly used way is to define the *Contribution to Value-at-Risk*, $c_i$, of the $i$'th position, $a_i$ to be as follows:

$$c_i = a_i MVaR_{\alpha,i}(\mathbf{a})$$

Conveniently the $c_i$ then sum to the overall VaR:

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i MVaR_{\alpha,i}(\mathbf{a}) = -\sum_{i=1}^{n}\left(a_i\mu_i + N^{-1}(1-\alpha)\frac{1}{\sigma}\left(a_i \sum_{j=1}^{n} a_j V_{ij}\right)\right)$$

$$\Rightarrow \sum_{i=1}^{n} c_i = -\left(\mathbf{a}.\boldsymbol{\mu} + N^{-1}(1-\alpha)\frac{\sigma^2}{\sigma}\right) = -\left(\mathbf{a}.\boldsymbol{\mu} + \sigma N^{-1}(1-\alpha)\right) = VaR_\alpha(\mathbf{a})$$

The property that the contributions to risk add to the total risk is a generic feature of any risk measure that is (first-order) homogeneous, a property that Value-at-Risk exhibits.