Showing that VaR is not coherent for exponentially distributed loss variables

[Nematrian website page: ERMMTVarNotCoherentForExponentialDistributions, © Nematrian 2015]

Please bear in mind that in general a multivariate distribution each of the marginal of which is exponentially distributed is not necessarily a member of the exponential family of (multivariate) distributions.

The simplest case to consider is where two loss variables come from the same exponential distribution with parameter λ. The probability density function of $Y = Y_1 + Y_2$ where Y_1 and Y_2 are independent random variables each coming from an exponential distribution with parameter λ is $f(y) = \lambda^2 y \exp(-\lambda y)$.

The VaR at confidence level α for Y_1, $VaR_\alpha(Y_1)$ is the same as $VaR_\alpha(Y_2)$ and is the value of q for which:

$$\int_q^\infty f_1(x)dx = 1 - \alpha \quad where\ f_1(x) = \lambda \exp(-\lambda x)$$

$$\Rightarrow \exp(-\lambda q) = 1 - \alpha \quad \Rightarrow \quad VaR_\alpha(Y_1) = VaR_\alpha(Y_2) = -\frac{\log(1-\alpha)}{\lambda}$$

The VaR at the same confidence level for Y is the value of q for which

$$\int_q^\infty f(x)dx = 1 - \alpha \quad where\ f(x) = \lambda^2 x \exp(-\lambda x)$$

Integrating by parts we have:

$$(1 + \lambda q)\exp(-\lambda q) = 1 - \alpha$$

The exponential distribution (with independent variables) is not elliptical so VaR's for it shouldn't be coherent. More specifically it should be possible to identify a suitable α for which VaR is not sub-additive, i.e. for which $VaR_\alpha(Y) > VaR_\alpha(Y_1) + VaR_\alpha(Y_1)$, as sub-additivity is one of the axioms required for coherence. Although there is no certainty that this is the case it is likely that such a situation will arise either where α is close to zero or when α is close to 1. The case where α is close to 1 does not look promising as the behaviour of $VaR_\alpha(Y)$ will tend to be dominated by the exponential term. So instead let us consider the case where when α is close to 0. We then have for $VaR_\alpha(Y)$:

$$(1 + \lambda q)\left(1 - \lambda q + \frac{\lambda^2 q^2}{2} + \cdots\right) \cong 1 - \alpha \quad \Rightarrow \quad \frac{\lambda^2 q^2}{2} \cong \alpha \quad \Rightarrow \quad VaR_\alpha(Y) \cong \frac{1}{\lambda}\sqrt{2\alpha}$$

However in these circumstances:

$$VaR_\alpha(Y_1) = VaR_\alpha(Y_2) = -\frac{\log(1-\alpha)}{\lambda} = \cong \frac{1}{\lambda}\alpha$$

So if α is small enough we will have $VaR_\alpha(Y_1) + VaR_\alpha(Y_2) < VaR_\alpha(Y)$ as desired to prove the conjecture.