Estimating operational risk capital requirements assuming data follows a triangular distribution (using maximum likelihood)

[Nematrian website page: ERMMTOperationalRiskCapitalTriangularDistributionMLE, © Nematrian 2015]

Suppose a risk manager believes that an appropriate model for a particular type of operational risk exposure involves the loss, $X \geq 0$, never exceeding an upper limit, $c > 0$, and the probability density function $f(x)$ taking the form:

$$f(x) = \begin{cases} a & \text{if } 0 \leq x < c/2 \\ b & \text{if } c/2 \leq x \leq c \\ 0 & \text{otherwise} \end{cases}$$

where $a > 0, b > 0, c > 0$ are all constant.

Suppose we want to estimate maximum likelihood estimators for a, b and c given losses of $X_1, \dots, X_n$, say and hence to estimate a Value-at-Risk for a given confidence level for this loss type, assuming that the probability distribution has the form set out above.

We note that $\int_0^b f(x)dx = 1$ for $f(dx)$ to correspond to a probability density function, so:

$$1 = \int_0^c f(x)dx = \int_0^{c/2} f(x)dx + \int_{c/2}^c f(x)dx = \frac{ac}{2} + \frac{bc}{2} = \frac{c}{2}(a+b)$$
$$\Rightarrow \quad a = \frac{2}{c} - b$$

Suppose the n losses, $X = (X_1, \dots, X_n)^T$, are assumed to be independent draws from a distribution with probability density function $f(x)$ and suppose n_1 of these losses are less $c/2$ and $n_2 = n - n_1$ are greater than $c/2$. The likelihood is then:

$$L(X) = \begin{cases} \prod_{i=1}^n f(X_i) = a^{n_1} b^{n_2} = \left(\frac{2}{c} - b\right)^{n_1} b^{n_2} \quad if\ all\ X_i \leq c \\ 0 \end{cases}$$

This will be maximised for some value that has $L(X) > 0$, i.e. has c at least as large as $max(X_1, \dots, X_n)$. In such circumstances the likelihood is maximised when the log likelihood is maximised which will be when $\frac{\partial}{\partial b} \log L(X) = 0$, i.e. when $\frac{\partial}{\partial b}\left(n_1 \log\left(\frac{2}{c} - b\right) + n_2 \log b\right) = 0$, i.e. when:

$$0 = -\frac{n_1}{\left(\frac{2}{c} - b\right)} + \frac{n_2}{b} = \frac{-n_1 b + n_2\left(\frac{2}{c} - b\right)}{\left(\frac{2}{c} - b\right) b}$$
$$\Rightarrow \quad b = \frac{2n_2}{c(n_1 + n_2)} = \frac{2}{c}\frac{n_2}{n} \quad and \quad a = \frac{2}{c}\frac{n_1}{n}$$

(assuming $a \neq 0$ and $b \neq 0$)

For these values of a and b the log likelihood is then:

$$n_1\left(\log\left(\frac{2}{cn}\right)+\log n_1\right)+n_2\left(\log\left(\frac{2}{cn}\right)+\log n_2\right)=n\log\left(\frac{2}{n}\right)-n\log c+n_1\log n_1+n_2\log n_2$$

In most circumstances this will be maximised when c is as small as possible, as long as c is still at least as large as $max(X_1,\dots,X_n)$ so the maximum likelihood estimators are:

$$\hat{c}=max(X_1,\dots,X_n),\quad \hat{a}=\frac{2}{\hat{c}}\frac{n_1}{n},\quad \hat{b}=\frac{2}{\hat{c}}\frac{n_2}{n}$$

However, it is occasionally necessary to consider the case where we select a $c>max(X_1,\dots,X_n)$ to that has a and/or b equal to zero.

To estimate a VaR at a confidence level α we need to find the value Y for which the loss is expected to exceed Y only $(1-\alpha)\%$ of the time, i.e. Y such that (if $Y\geq c/2$):

$$1-\alpha=\int_Y^c f(x)dx=(c-Y)b$$
$$\Rightarrow\quad Y=c-\frac{1-\alpha}{b}$$