Deriving the Black-Scholes Option Pricing Formulae using Ito (stochastic) calculus and partial differential equations

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The following partial differential equation is satisfied by the price of *any* derivative on S, given the assumptions underlying the Black-Scholes world:

$$-ru + u_t + (r-q)Su_S + \frac{\sigma^2 S^2}{2} u_{SS} = 0$$

This partial differential equation is a second-order, linear partial differential equation of the *parabolic* type. This type of equation is the same as used by physicists to describe diffusion of heat. For this reason, Gauss-Weiner or Brownian processes are also often commonly called *diffusion* processes.

If r, q and σ are constant then we can solve it by transforming it into a standard form which others have previously solved, namely (for some constant c):

$$\frac{1}{c^2}\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial x^2} \quad \text{i.e. } w_y = c^2 w_{xx}$$

This can be achieved by replacing u by w where $w = ue^{r(T-t)}$ (as long as r is constant) and by making the following double transformation (assuming that r, q and σ are constant):

This transformation removes one of the terms in the partial differential equation:

$$y = T - t \qquad x = \frac{\log(S)}{\sigma} + \frac{r - q - \sigma^2/2}{\sigma}(T-t) = \frac{\log(S)}{\sigma} + \frac{r - q - \sigma^2/2}{\sigma} y$$

The partial differential equation then simplifies to $w_y = c^2 w_{xx}$, with $c = 1/\sqrt{2}$. Prices of different derivatives all satisfy this equation and are differentiated by the imposition of different *boundary conditions.* A common tool for solving partial differential equations subject to such boundary conditions is the use of *Green's* functions. This expresses the solution to a partial differential equation given a general boundary condition applicable at some boundary $B(z)$, formed say by the curve $x = \bar{x}(z), y = \bar{y}(z)$, as an expression of the following form, in which G is called the Green's function for that partial differential equation:

$$v(x,y) = \int_B v_0(z) G(x,y,\bar{x}(z),\bar{y}(z))\, dz$$

The Green's function for $w_y = c^2 w_{xx}$ where c is constant is:

$$G(x,y,\bar{x},\bar{y}) = \frac{1}{2c\sqrt{\pi}} \frac{e^{-(x-\bar{x})^2/\left(4c^2(\bar{y}-y)\right)}}{\sqrt{y-\bar{y}}}$$

If the boundary condition is expressed as $w(x,0) = w_0(x)$ at $y = 0$, where $w_0(x)$ is continuous and bounded for all x, then the solution is:

$$w(x,y) = \frac{1}{2a\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{w_0(z)}{\sqrt{y}} e^{-(z-x)^2/(4c^2y)} dz$$

For a European call option, with strike price K, we have, after making the substitutions described above, $u(S,T) = \max(S-K,0) \Longrightarrow w_0(x) = \max(e^{\sigma x} - K, 0)$. After some further substitutions we find that this implies that:

$$u(S,t) = Q(1,K,S,T-t) - KQ(0,K,S,T-t)$$

where:

$$Q(k,K,S,y) = S^k e^{-kqy-r(1-k)y+\sigma^2 y(k^2-k)/2} N(H)$$
$$H = \frac{\log(S/K) + (r - q - \sigma^2/2 + k\sigma^2)y}{\sigma\sqrt{y}}$$

Substituting $k = 0$ and $k = 1$ into the formula for $Q(k,K,S,y)$ recovers the Black-Scholes formulae, e.g. for a call option:

$$C(S,t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

Where

$$d_1 = \frac{\log(S/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and $N(z)$ is the cumulative Normal distribution function, i.e.

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$