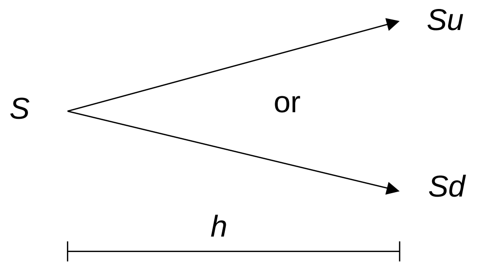
Deriving the Black-Scholes Option Pricing Formulae using the limit of a suitably constructed lattice

Suppose we knew for certain that between time $t-h$ and t the price of the underlying could move from S to either Su or to Sd, where $d<u$ (as in the diagram below), that cash (or more precisely the appropriate risk-free asset) invested over that period earns an interest rate (continuously compounded) of r and that the underlying (here assumed to be an equity or an equity index) generates income, i.e. dividend yield, (continuously compounded) of q.

Diagram illustrating single time-step binomial option pricing



Suppose that we also have a derivative (or indeed any other sort of security) which (at time t) is worth $A=V(Su,t)$ if the share price has moved to Su, and worth $B=V(Sd,t)$ if it has moved to Sd.

Starting at S at time $t-h$, we can (in the absence of transaction costs and in an arbitrage-free world) construct a hedge portfolio at time $t-h$ that is guaranteed to have the same value as the derivative at time t whichever outcome materialises. We do this by investing (at time $t-h$) fS in f units of the underlying and investing gS in the risk-free security, where f and g satisfy the following two simultaneous equations:

$$fSue^{qh}+gSe^{rh}=A=V(Su,t)$$
$$fSde^{qh}+gSe^{rh}=B=V(Sd,t)$$

Hence:

$$fS=e^{-qh}\frac{V(Su,t)-V(Sd,t)}{u-d}\quad gS=e^{-rh}\frac{-dV(Su,t)+uV(sd,t)}{u-d}$$

The value of the hedge portfolio and hence, by the principle of no arbitrage, the value of the derivative at time $t-h$ can thus be derived by the following *backward equation*:

$$V(S,t-h)=fS+gS=\frac{e^{(r-q)h}-d}{u-d}e^{-rh}V(Su,t)+\frac{u-e^{(r-q)h}}{u-d}e^{-rh}V(Sd,t)$$

We can rewrite this equation as follows, where $p_u=\left(e^{(r-q)h}-d\right)/(u-d)$ and $p_d=\left(u-e^{(r-q)h}\right)/(u-d)$ and hence $p_u+p_d=1$.

$$V(S,t-h)=p_ue^{-rh}V(Su,t)+p_de^{-rh}V(Sd,t)$$

Assuming that the two potential movements are chosen so that p_u and p_d are both positive, i.e. with $u > e^{(r-q)h} > d$ then p_u and p_d correspond to the relevant risk neutral probabilities for the lattice element. Getting p_u and p_d to adhere to this constraint is not normally difficult for an option like this since $e^{(r-q)h}$ is the forward price of the security and it would be an odd sort of binomial tree that did not straddle the expected movement in the underlying.

In the multi-period analogue, the price of the underlying is assumed to be able to move in the first period either up or down by a factor u or d, and in second and subsequent periods up or down by a further u or d from where it had reached at the end of the preceding period. u or d can in principle vary depending on the time period (e.g. u might be size u_i in time step i, etc.) but it would then be usual to require the lattice to be *recombining*. In such a lattice an up movement in one time period followed by a down movement in the next leaves the price of the underlying at the same value as a down followed by an up. It would also be common, but again not essential (and sometimes inappropriate), to have each time period of the same length, h.

By repeated application of the backward equation referred to above, we can derive the price n periods back, i.e. at $t = T - nh$, of a derivative with an arbitrary payoff at time T. If u, d, p_u, p_d, r and q are the same for each period then:

$$V(S, T-nh) = e^{-rnh} \sum_{m=0}^{n} \binom{n}{m} p_u^m p_d^{n-m} V(Su^m d^{n-m}, T)$$

where:

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

This can be re-expressed as an *expectation* under a risk-neutral probability distribution, i.e. in the following form, where $E(X|I)$ means the expected value of X given the risk neutral measure, conditional on being in state I when the expectation is carried out:

$$V(S,t) = E\left(e^{-r(T-t)} V(S,T) \middle| S_t\right)$$

Suppose we have a European-style put option with strike price K (assumed to be at a node of the lattice) maturing at time T and we want to identify its price, $P(S,t)$ prior to maturity, i.e. where $t < T$. Suppose also that r and q are the same for each time period. The price of the option at maturity is given by its payoff, i.e. $P(S,T) = \max(K - S, 0)$ where $K = S_0 u^{m_0} d^{n-m_0}$ say for some m_0 (here S_0 is the price ruling at time $t = 0$ used to construct the first node in the lattice). Applying the multi-period pricing formula set out above, we find that the price of such an option at time $t = T - nh < T$ in such a framework is as follows, where $B(x,n,p)$ is the binomial probability distribution function, i.e. $B(x,n,p) = \sum_{m=0}^{x} \binom{n}{m} p^m (1-p)^{n-m}$, bearing in mind that $p_u + p_d = 1$:

$$P(S, T-nh) = e^{-rnh} \sum_{m=0}^{m_0} \binom{n}{m} p_u^m p_d^{n-m} (K - S_0 u^m d^{n-m})$$
$$\Rightarrow P(S,t) = e^{-r(T-t)} K B(m_0, n, p) - e^{-q(T-t)} B\left(m_0, n, \frac{up_u}{up_u + dp_d}\right)$$

Suppose we define the *volatility* of the lattice to be $\sigma = \log(u/d)/\left(2\sqrt{h}\right)$ and suppose too that this is constant, i.e. the same for each time period. Then if we allow h to tend to zero, keeping σ, t, T

etc. fixed, with $u/d \longrightarrow 1$ by, say, setting $\log(u) = \sigma\sqrt{h}$ and $\log(d) = -\sigma\sqrt{h}$, we find that the above formula and hence the price of the put option tends to:

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1)$$

where

$$d_1 = \frac{\log(S/K) + (r - q + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 = d_1 - \sigma\sqrt{T - t}$$

and $N(z)$ is the cumulative Normal distribution function, i.e.

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$

The corresponding formula (in the limit) for the price, $C(S,t)$ of a European call option maturing at time T with a strike price of K can be derived in an equivalent manner as:

$$C(S,t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

This formula can also be justified on the grounds that the value of a combination of a European put option and a European call option with the same strike price should satisfy so-called put-call parity, if they are to satisfy the principle of no arbitrage, i.e. (after allowing for dividends and interest):

$$stock + put = cash + call$$
$$\Rightarrow Se^{-q(T-t)} + P = Ke^{-r(T-t)} + C$$

Strictly speaking, these formulae for European put and call options are the *Garman-Kohlhagen* formulae for dividend bearing securities and only if q is set to zero do they become the original *Black-Scholes* option pricing formulae, although in practice most people would actually refer to these formulae as the Black-Scholes formulae, and call a world satisfying the assumptions underlying these formulae as a 'Black-Scholes' world. The volatility σ used in their derivation has a natural correspondence with the volatility that the share price might be expected to exhibit in a Black-Scholes world.